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A Problem Suggested by Celtic, Art ***American Mathematical Monthly***, V90, No.2 Feb 1983, pp 122-125, Solution text in Dec. 1983 issue included at end of this reprint. This paper was edited by Prof. Richard K. Guy who generously accepted the paper for publication. Several responses followed and one fellow was looking at this as part of his Phd thesis. It is now presented by me as ECCurves with a program on my website Puzzleatomic.com
http://www.puzzleatomic.com/EC%20CURVE%20SYSTEM.htm
that can create mathematical art somewhat in the spirit of the Ancient Celts. They believed that a closed loop represented the life and death loop of each living thing. Much interesting math is hidden in these product patterns.

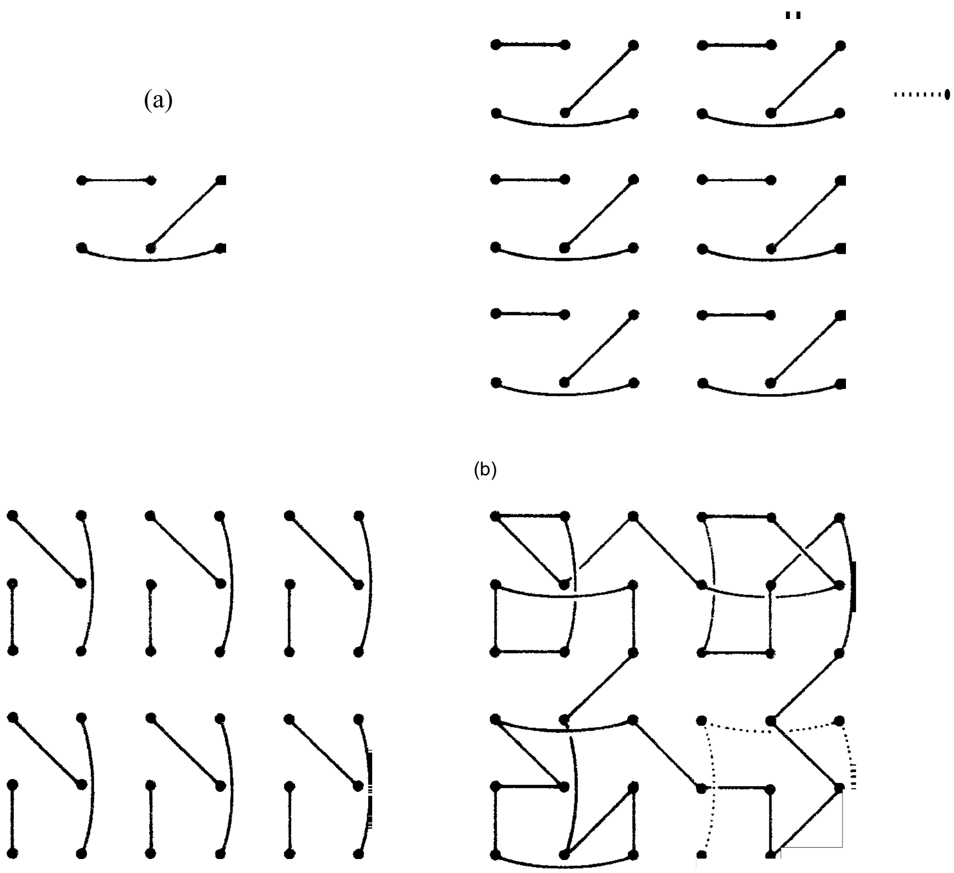
A Problem Suggested by Celtic Art

Douglas A. Engel, 2935 **W.** *Chenango, Englewood, CO 8011 (my old address)*

Ancient Celtic artists sometimes used the following device to produce braided patterns, although they did not exactly follow the instructions given here, which have been crystallized into a precise problem.

Draw a graph on a rectangular array of m rows of n points, where m and n are co-prime and just one of m and n is even. The $mn/2$ edges of the graph form a matching. or regular graph of valence one. In Figure 1a, $m = 2$ and $n = 3$. Next arrange mn copies of the graph in n rows of m, to produce a square array of mn rows of mn points connected in pairs by $(m^2)(n^2)/2$ edges. as in Figure lb. Thirdly rotate this square array through 90° about its centre. as in Figure l(c). Finally, superimpose this rotation onto the original square array to give a pattern as in Figure 1d This is a regular graph of valence 2 and so consists of a number of disjoint cycles. For example. the 36 edges in Figure lrd) form a 4-cyclc (shown dotted) and a 32-cycle .

The Celtic artists preferred their finished product to be a single $(m^2)(n^2)$ cycle or **hamilton cycle.**



(c) **Fig. 1** **(d)**

Problem 1. What are the conditions on the original matching in order that the final pattern is a (single) Hamilton cycle?

If m = l and n = 2*k*, then we always generate *k 2* 4-circuits. so that there is a Hamilton cycle only in the trivial case *m* = I, n = 2. Some solutions for the case *m* = 3, n = 4 arc shown in Figure 2. Mr. Engel has sent this editor a model of the fourth of these. made from wire and perspex; It is now on display in his Department.]

It may be possible to answer the next two problems by using a computer.

Problem 2. Find all matchings which lead to a Hamilton cycle when m = 3 and n = 4.

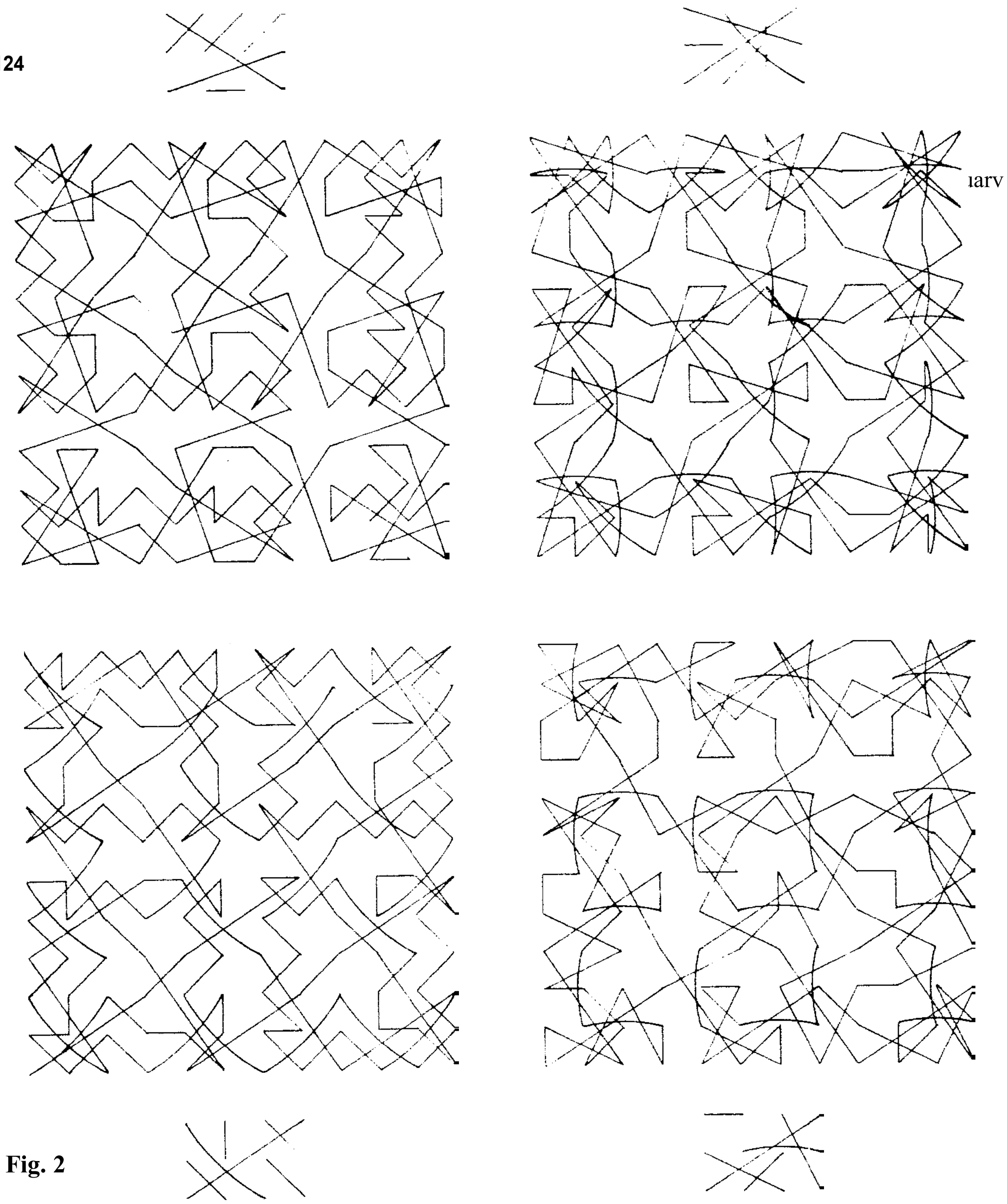


Fig. 2

Problem 3. Find such a matching when $m = 4$ and $n = 5$.

Problem 4. Are there values of m, n with $(m, n) = 1$ and mn even for which no matching generates a Hamilton cycle?

As m and n increase, solutions seem harder and harder to find. but this is a common phenomenon in combinatorial searches. The number of different matchings is something like *(mn/e)*"'' 2 so that. Reven if the number of solutions increases exponentially, sav like (*mn/e^(mn*/2) *[Richard Guy's very interesting equation]* for some constant c, then the chance of finding one at random is ((c^2)e/ mn)^mn/2 which soon becomes hopelessly small as m and n increase.

Problem 5. Find bounds. or even an asymptotic formula, for the number of distinct solutions for a given m and n.

If $m = 2$ and n is odd, then solutions arc known (Figure 3 l which have a fourfold symmetry of rotation (through 90 degrees).

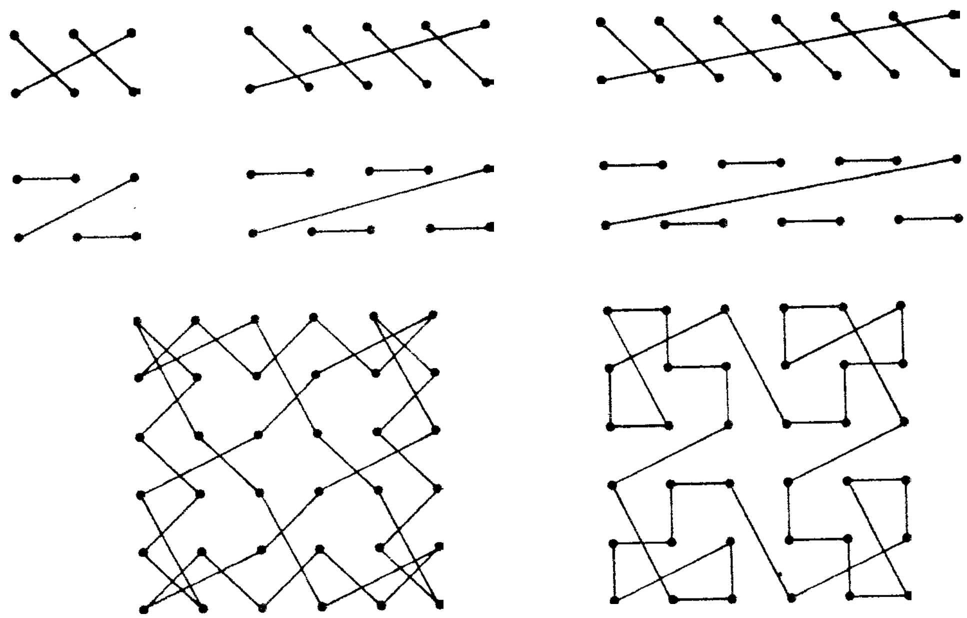


Fig. 3

Problem 6. Find such symmetrical solutions for $3 <= m < n$.

Similar problems may be formulated for the case where the square array is reflected as well as rotated before being superimposed on the original. We can also ask how to formulate corresponding problems in three dimensions, based on crystallographic lattices.

Reference: **George Bain**, ***Celtic Art*** Dover, New York, 1973

MISCELLANEA (included by R. K. Guy)

93. I once heard the great Richard Owen [the biologist] say... that he would like to see *Homo Mathematicus* constituted into a distinct subclass, thereby suggesting to my mind sensation, perception, reflection, abstraction, as the successive stages or phases of protoplasm on its way to being made perfect in Mathematicised Man

-J. J. Sylvester, 1870. (Reprinted in *Collected Mathematical Papers,* vol. 2, Cambridge, 1908, p 652.)

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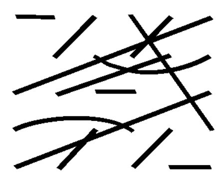
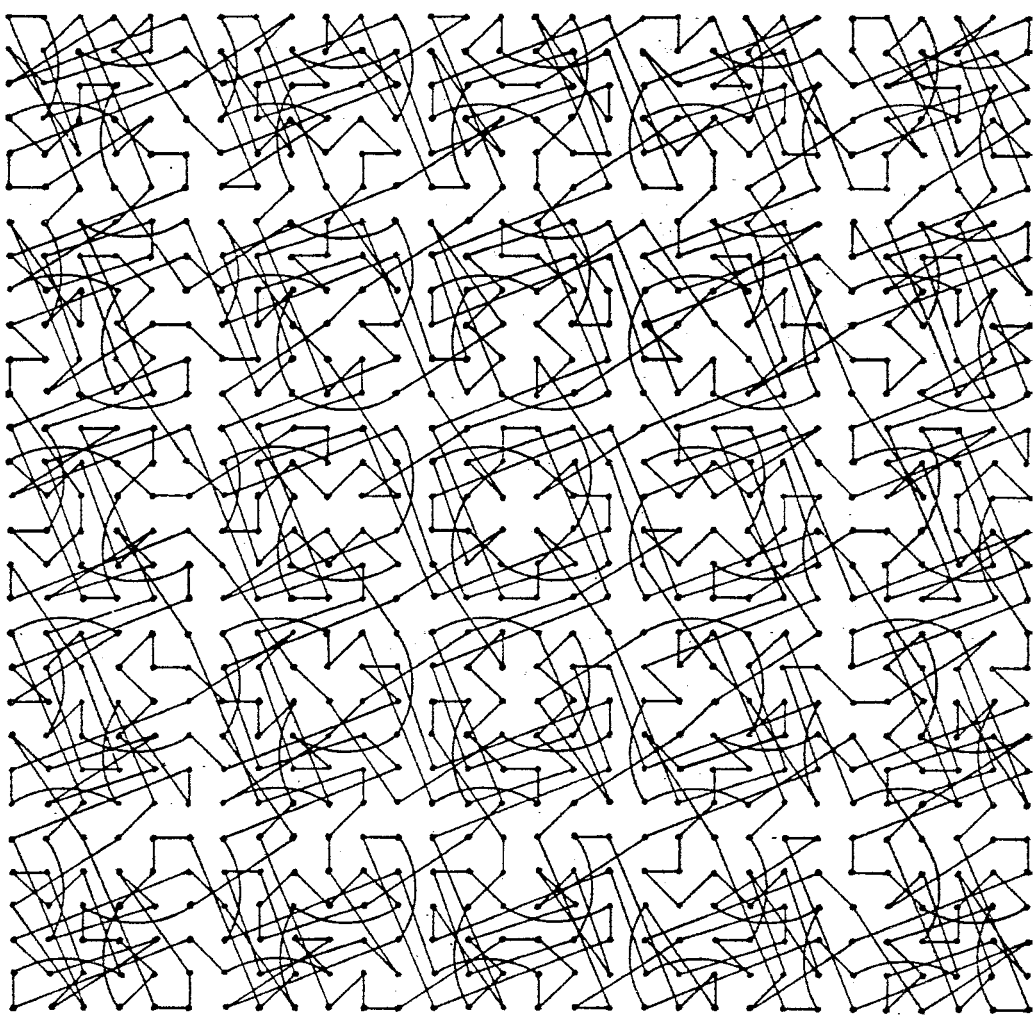
Engel's "Celtic Art" problem [1983, 122] stimulated John P. Robertson to use a computer to find the 42 essentially different 3 X 4 matchings which lead to Hamilton cycles. There are two different 2 x 3 matchings, each of which leads to a pattern with 4-fold rotational symmetry.

These were depicted earlier [1983, 125]. Of the eight 2 X 5 matchings which generate Hamilton cycles, all but one have such symmetry. But none of the 42 (3 X 4) generated patterns is symmetrical. Engel had earlier noted that this rotational symmetry only seems to occur if mn is singly even, not when *mn* is a multiple of 4. Is this a theorem, or haven't we searched far enough? [Stop press! Robertson writes that he has a proof.] Robertson has found symmetrical patterns generated by 3 x 10, 5 x 6 and 6 x 7 matchings, but none of the seven 4 X 5 matchings that he found leads to a symmetrical Hamilton circuit. Figure 2(a) is a 5 X 6 matching which gives the



(a)



(b)

FIG. 2 The matching (a) generates the Hamilton circuit (b): symmetrical pattern shown in Figure 2(b). A reference omitted from the original article is:

Doug Engel, Wallpaper Rings, Journal of Recreational **Math.,** 13 (1980-81) 7-9