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How a Flexible Tetrahedral Ring Became a Sphinxx

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A regular tetrahedron consists of four plane equilateral triangles arranged symmetrically about a center. Regular tetrahedrons cannot be stacked together to perfectly fill three dimensional Euclidean space. However, there are other types of tetrahedrons which can perfectly fill three dimensional Euclidean space.

One of these is a tetrahedron which is built up of four congruent isoscles triangles and has two opposite dihedral angles equal to 90° and four dihedral angles equal to 60°. The tetrahedron can be made from the pattern shown in Figure 1. This basic tetrahedron will be called a *link* in this article. The edge of a link

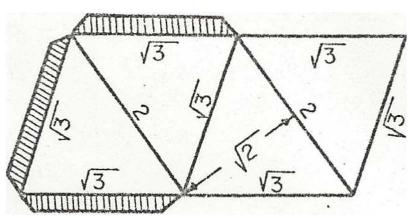

FIGURE **1**

at which a dihedral angle of 90° is formed will be called a *hinge axis.* In Figure 2 two links are shown taped together at a hinge axis. They are free to rotate about their hinge axes 180° with respect to one another.

The rest of the discussion will be devoted to describing how the links may be used to create different flexible chains and rings. It will then attempt to describe how one of these rings

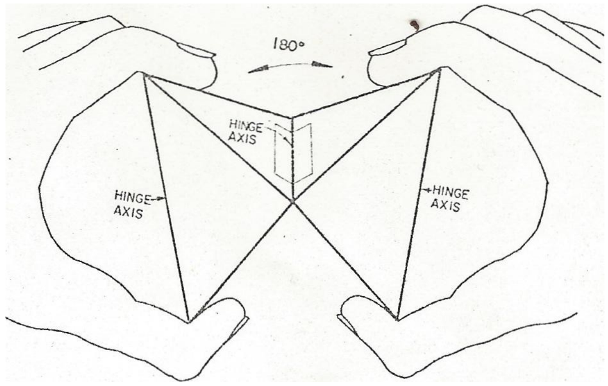

FIGURE 2

can interlock to form solids of rather exotic character which will perfectly stack together to fill three dimensional Euclidean space.

The hexaflexagons discovered in 1939 by Arthur H. Stone were the original motivation for the construction of the solid *[texahedrons* in this article. Martin Gardner's article in *Scientific American* [I] gives .an excellent description of some of the hexaflexagons.

All of the results beyond the link in this discussion were originally arrived at by trial and error. There are no straight forward mathematical methods of doing it as far as is known. What was discovered was arrived at by following a random trail.

If eight links are taped together at their hinge axes, they form an eight link *chain.* This chain may now be connected into a *ring* by . using the remaining two hinge axes at the head and tail ends of the chain.

It is not possible to give the ends of the eight link chain a full twist before connecting into a ring. It is possible to continuously turn the eight link ring in the same direction about a circular axis formed by a circle passing through the points of the centers of the eight links.

When 16 links are connected into a chain, a 360° twist can be given to the chain before connecting it into a"' ring. When 24 links are connected into a chain, two 360° twists are possible before connecting it into a ring. Both the twisted ring of 16 links and the twisted ring of 24 links will undergo a cycle of flections so that they appear to turn inside out. In one cycle of flections, a ring returns to its starting position.

In order to effect a single flection, a set of links of a ring must be rotated with respect to the rest of the ring. This rotation may be of 90° or 180° and always occurs about two hinge axes which are in line.

The ring does not really turn inside out if you try to think of it as a solid enclosing a three dimensional space. An imaginary axis can be drawn connecting the centers of all the links in the ring called the ring axis. A ring is *flexible* if the links of the ring can be made to rotate continuously in the same direction about the ring axis.

A chain of 32 links can be given three full twists before connecting into a ring. It is also flexible as defined above. A chain of 40 links can be given four full twists before connecting into. a ring. It is not flexible in three dimensional Euclidean space. This author has conjectured that it would be flexible in a four dimensional space. It can be forced to complete a cycle of fleetions if one is careful not to tear it apart.

We will now return to the ring of 24 links with two 360° twists. Figure 3 shows this ring undergoing a complete cycle of flections. Nine separate rotations of various parts of the ring are required before the cycle is complete. Because of the twists in the ring only certain motions are allowed. The twists appear to restrict the freedom of the ring. A flexible ring can best be flexed by taking symmetrical sets of opposed parts for each rotation. The twist given to the ring makes it asymetrical as a whole in three dimensional Euclidean space.

Consider position 1 or 9 in Figure 3 for the 24 link ring.
In this position all the links have one of their faces adjacent to a face of another link. This is called the *closed angle position.* As far as is known no other flexible ring has a closed angle position.

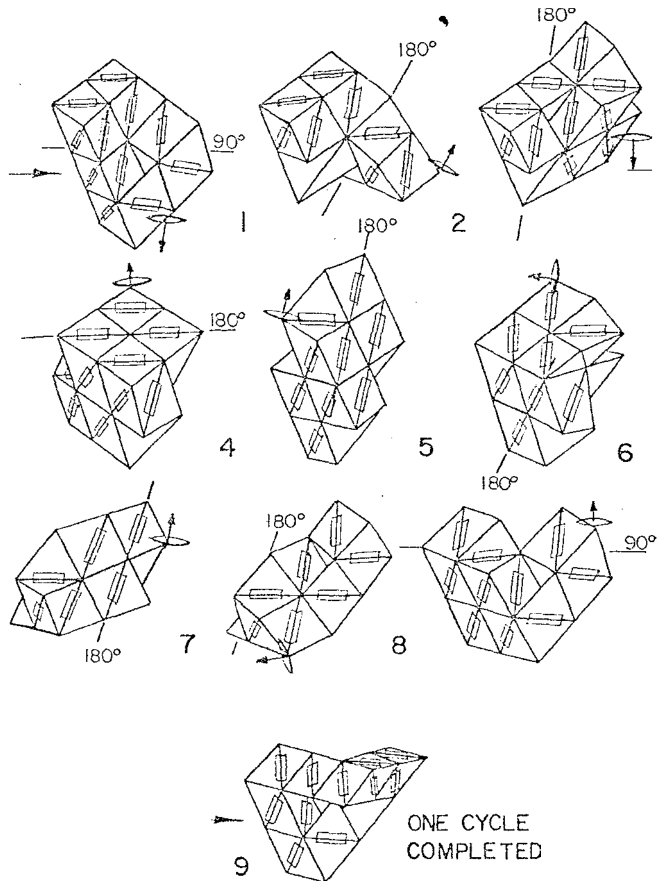

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FIGURE 3

A wood rod that has an equilateral triangle for a cross section may be cut into congruent pieces as shown in Figure 4. Each of these pieces is equivalent to a chain of six links with a 180° twist. Figure *5* shows how four of these rods may be glued together to form a solid which duplicates position 9 of the 24 link ring shown in Figure 3. In this position, the ring has two triangular holes which allow two of the solid rings made of wood to interlock in various ways.

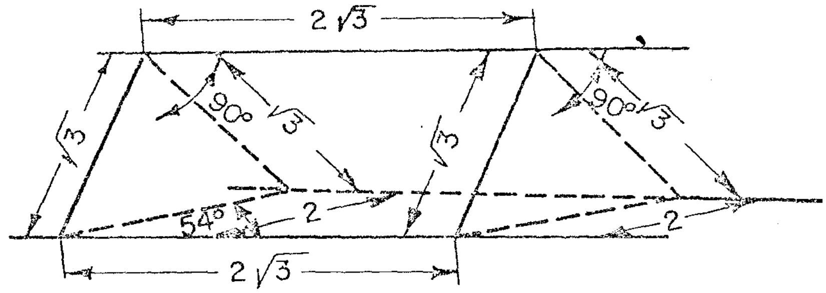

FIGURE 4

In what follows the 24 link ring in its closed angle position will be called a *sphinxx.* * The sphinxx should be considered as a new solid derived from the 24 link ring. The author named it a sphinxx because of its puzzling property of forming larger rings.

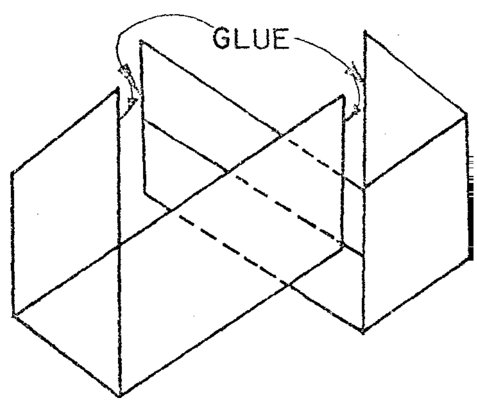

FIGURE 5

* It is spelled with two x's to distinguish it from the legendary Sphinx.

Two sphinxx may interlock in four different ways as shown in Figure 6. One of the two-sphinxx chains is actually a ring (number 4 in Figure 6). This two sphinxx ring has, among other properties, the ability to perfectly fill three dimensional Euclidean space in a number of different ways. Figure 7 illustrates how a "plane" of these rings can be stacked together. These "planes" may then be stacked one atop the other to completely fill three space.

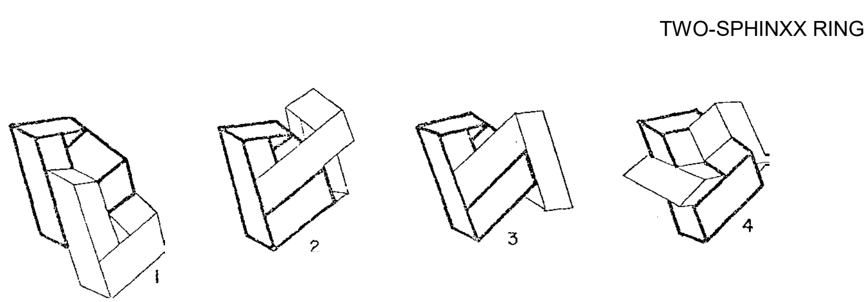

FIGURE 6

The alert reader will now begin to conjecture about forming more complicated structures by connecting a greater number of sphinxxs into rings. Six sphinxx will combine together in three different ways to form rings. This is shown in Figures 8, 9, and 10. Two of these contain a 360° twist. This. can be shown by making a ring out of rubber links and then cutting it apart at one link and straightening it into a chain. The other (Figure 10) has a zero twist. All three can perfectly fill three dimensional Euclidean space by stacking the rings into a row as shown in Figures 8, 9, and 10. Rows of rings may then be placed side by side to form a continuous "plane" of rings. These "planes" can be stacked as before to completely fill three space.

With this the discussion is complete. The mystery of why the sphinxx form complicated rings which happen to fill three space remains. Part of the mystery may be connected with a property discussed in a previous paper about a twisted ring made of eight sphinxx [2]. Another part of the mystery deals with something very difficult to illustrate in a picture. It is a generalized solid made of sphinxx. The solid itself may be 11lken apart into a set of complete sphinxx rings. It is therefore really composed of a set of interknotted sphinxx rings.

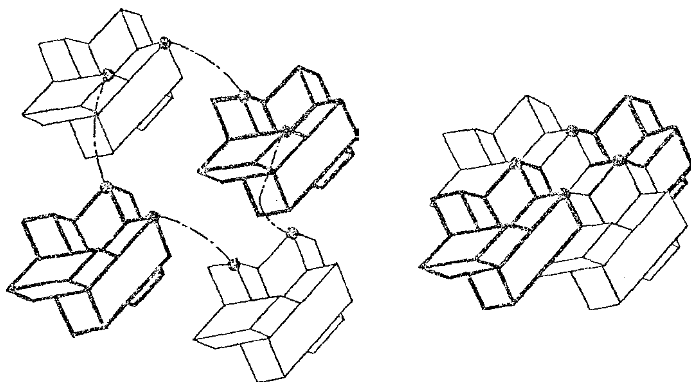

FIGURE 7

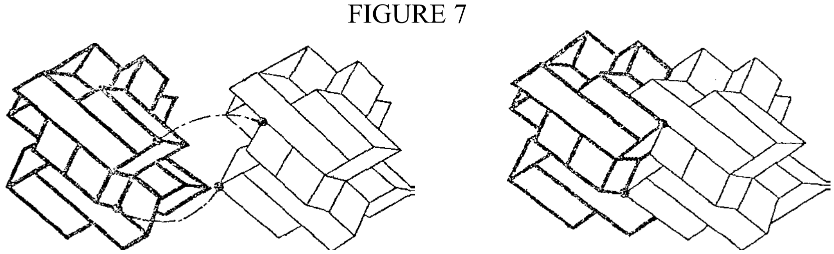

FIGURE 8

This solid will probably never be drawn or illustrated schematically because there is no need to. Several of them have been constructed by the author. This generalized super-sphinxx can

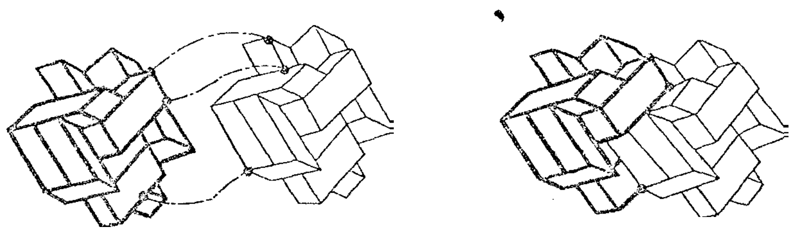

FIGURE 9

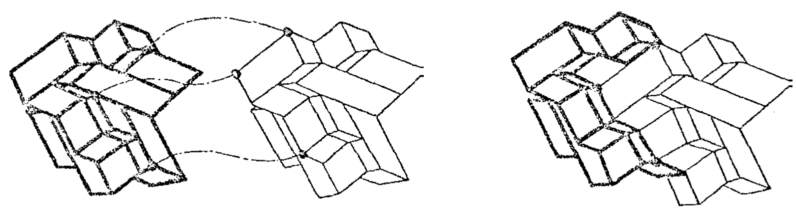

FIGURE 10

be described mathematically with a set of simple equations containing three variables. The rings which make up a super-sphinxx · and their mutual relationships appear to be closely allied to number theory (prime numbers, factored numbers, and congruences.)

Perhaps in a future discussion, the properties of the supersphinxx can be developed. The sphinxx may then in its own way have added to the interest of the delightful field of number theory.

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