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(This represents a portion of research on all sorts of twisted topological structures, It is unique in first considering the flattening of the structures to go from 3D to 2D. DAE)

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Hybrid Flexahedrons

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This article is more concerned with making flexagons than flexahedrons. The reader will want to construct the flexagons derived from the flexahedrons because flexagons are easier to make and flex. General instructions for making flexagons are covered in [3, 4]. The flexahedra and flexagons discussed below exhibit many of the properties of a more general type of flexahedron [1, 3]. To begin with, I was struck by the possibility of combining regular octahedra with regular tetrahedra (of equal edges) because the two fill space in combination, but not alone.

The simplest symmetrical ring found is that in Figure 1 consisting of four tetrahedra, connected alternately as shown, with four octahedra. A glance at Figure 1 will show that all the hinge axes lie in parallel planes. This means that a larger ring will probably need quite a number of elements to get many twists between the ends before connecting them. By trial and error it was found that a minimum of 32 elements is needed to get a 360° twist between the ends of a ring before connecting. With more trial and error a general formula for the number of twists in such a ring was found to be

$$\frac{n}{16} - 1, \qquad (1)$$

where n is the number of elements and 360° equals one twist.

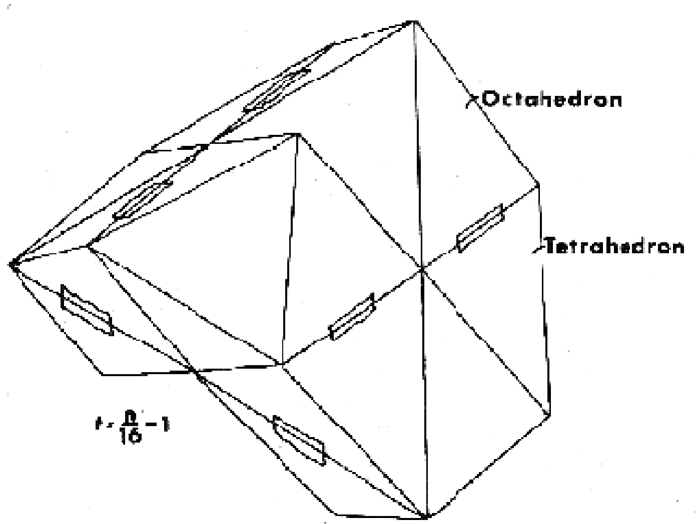

FIGURE 1

Analysis of the ring in Figure 1 elicited some very interesting questions. For instance, there appears to be no reason why one could not flatten any such ring along one or two or three of its axes simultaneously and get a flexatope. This has been done in Figure 2. It will be seen that as long as the flattening in any set of alternate parallel planes of elements is equal and orthogonal, any combination of flattenings should result in a flexible ring. The interesting feature of such flattening, to be discussed in this article, is what happens when various axes are flattened to zero. In other words, what type of flexatopes will result given a specific combination of infinite flattenings or a specific hybrid flexahedron? What will be the maximum twist of any such ring, and which rings are most interesting?

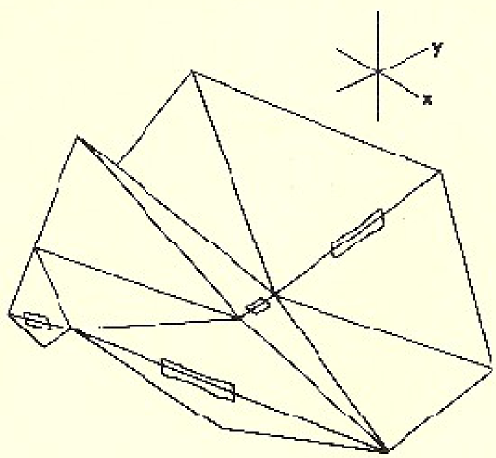

FIGURE 2

In Figure 3 the y distance has been made equal to zero. The resulting flexagon is composed of triangles (flattened tetrahedrons) and squares (flattened octahedrons). It is difficult to construct such a flexagon because some of the axes such as A and B in Figure 3 have to be connected at a vertex of a triangle and a square and should still allow rotation only about the y axis. However, with a little ingenuity it is possible to make good workable models of tin or cardboard. It was found, again by trial and error, that the formula for the number of twists in such a ring—which we shall call Y° for short—is

$$\frac{n}{8} - 1. \tag{2}$$

The increase in twist over the ring in Figure 1 is due to the fact that elements can occupy positions on either side of the plane that lies on the x and z axes. If the reader will make some workable models he should be able to verify quickly this formula for twist in $Y°$, and should discover how to flex various rings. Only rings that have maximum twist and are flexible should be considered as actual flexatopes.

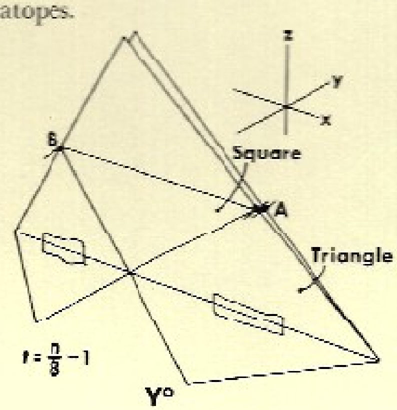

FIGURE 3

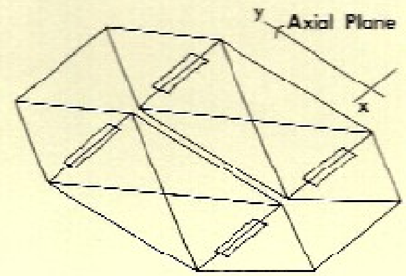

FIGURE 4

The most interesting flattening transformation occurs when the z axis is made equal to zero. Figure 5 shows the ring of Figure 1 with the z axis equal to zero. The resulting structure ($Z°$ for short) lies in the xy plane. $Z°$'s are made of large and small squares which are flattened octahedra and tetrahedra, respectively. It should be easy to make and flex any size model you desire out of cardboard and good adhesive tape. Figure 5 shows a $Z°$ of eight elements with two different transformations along both the x and y axes. It will be found that in any size ring of $8n$ elements only two different transformations are possible along each axis and they must be uniform so that hinge axes remain in line and orthogonal.

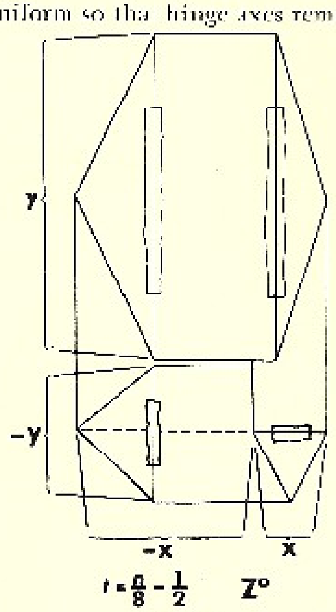

FIGURE 5

The formula for the number of twists in flexible rings of the Z° type was found to be

$$\frac{n}{8} - \frac{1}{2} \tag{3}$$

In actual fact a much greater twist is possible in these rings but it renders them completely inflexible. The increase in twist is due to the fact that elements can be on either side of the axial plane. The ring in Figure 5 is shown with a half, or 180°, twist, recycling with eight separate rotations of three sets of elements at a time. The Z° in Figure 4 is, strictly speaking, not a Z° since it does not have a maximum twist. It exhibits properties very similar to the ring in Figure 1.

Larger size Z° rings must always contain $8n$ elements. Of these, it is obvious that half must be the large squares and half the small squares connected together alternately. It is fun to make and flex larger-sized rings, especially because they exhibit a property which is lost in most flexagons such as the hexaflexagons. This property is one which causes the ring to cover more area as it becomes larger. If you make a ring of 16 elements you should be able to see that it is impossible to connect the ring in such a way that it covers only as much area as the 8-element ring in Figure 4. This unique property makes it possible to find ways of flexing rings of 24 or more elements that follow a sequence of geometrical shapes before recycling. Some of these ways will be quicker and more efficient than others. In particular, it is interesting to find the least efficient way that makes no backtracks. The most efficient way may produce few shapes and may recycle more quickly, thus being less interesting.

Using formula (3), the number of twists in a 16-element Z° ring is 1.5. Such a ring is shown going through 1/16 cycle in Figures 6 through 11. Fifteen similar operations are needed to make the ring recycle and bring the number 1 back to the position in Figure 6. You may be able to follow every rotation and new position in your mind. If you can do this you may have no need to construct any of the models for yourself. However, it is fun to introduce a 16-element model of a Z° ring at a party, or at an odd moment when having lunch with a couple of friends. If you do this you will be amazed at the number of different approaches people will take to flex the ring and how many different ways there seem to be. But the area-covering nature of these rings makes it possible to select different sets of elements to rotate and will thus make the flexing process seem quite different when it is actually the same. After watching people take it through its various positions you might become quite proficient at orienting the ring in your mind and making it flex in your imagination. From this beginning the enthusiast may mentally construct other rings and have quite a time flexing them. At any rate, I have found it to be one of the most creative and challenging disciplines for studying space movements.

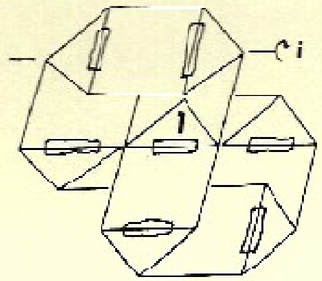

FIGURE 6

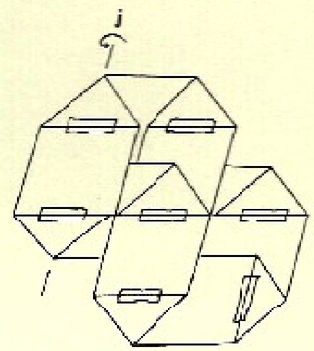

FIGURE 7

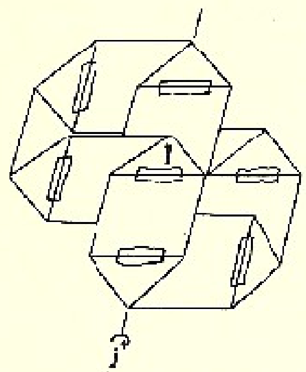

FIGURE 8

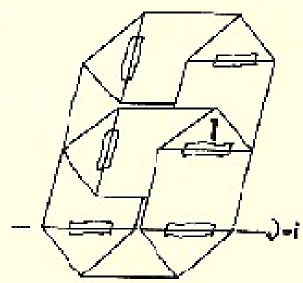

FIGURE 9

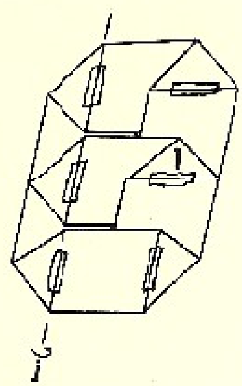

FIGURE 10

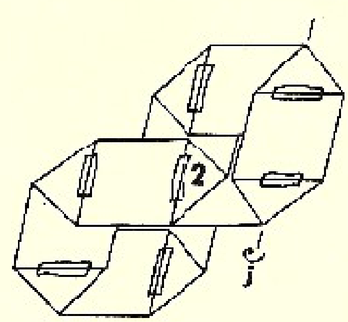

FIGURE 11

In flexing a Z° ring only two sets of parallel axes are brought into play. These may be called i and j and may be thought of as representing unit vectors in two dimensions. A flexing ring goes through time so an extra dimension is added when flexing. Even more dimensions may be added, hypothetically, by rotating a set of elements only part of the way and rotating the connected set the rest of the way to complete a rotation. This produces any desired change in the direction of a set of axes in space.

Thus one can create a set of movements that span a space somewhat similarly to the way a specific number of vectors span a space according to their matrix representation. From the i-j plane rings one might conclude that a more general type ring exists where one does not have to make any partial rotations. This type ring should be extendable to all dimensions, ideally. Such a ring can be made from tetragonal sphenoids with two opposite angles of 90°. It can be described as a set of i, j, and k unit vectors and rotations occur about i, j, and k axes. The resulting system fits the idea of Lie algebras [5] and there seems to be some connection with subatomic particles. Hence, the study of rotations in space as they occur in flexatopes may not be entirely useless.

The interested flexagonist may want to make much larger Z° rings and investigate their properties. Perhaps you can create a game of a ring with a fairly large number of elements laid out on a table where a set of players take turns making a given number of rotations of the parts of the ring. If someone has to backtrack it counts against him. But if someone achieves a certain planned pattern, he gains a point. This could be a very intriguing game if given the right set of rules and using a large enough ring.

It is curious that we have been able to flatten these hybrid flexahedrons in so many ways and create so many different flexagons. What happens when we flatten two axes at once? It is obvious that when either x and z, or y and z are made zero a similar 'flexaline' will result because of symmetry about the z axis. What is the difference between this type 'flexaline' and one of the type $X^\circ Y^\circ$? What are the formulas for the number of twists in flexalines? Can you construct one with lengths of wire? Since we are changing dimensions once again the formula for the number of twists probably will be $n/4$ plus or minus a small constant. This line of conjecture raises an interesting question: What would be the formula for the number of twists in a 'flexapoint'? This is a flexatope where all axes of a hybrid flexahedron have been made zero in some given order. Can a flexapoint even exist? The problem is left to the interested reader.

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