All Circle Link Systems

March 17, 2015 A report on investigations of linkages of solid circles.

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All circle links, ACL's are made by linking solid circular tori together. They can be viewed mathematically as being a topic somewhere between topology and geometry. The main reason for this report is to try to convey the fun and mental recreation that can be had working with ACL's. Because we are using solid, flat 3D circles and not topological strings the term projection is not used. Instead we use the term display or display position to avoid mathematical confusion since solid circles cannot be made flat.

The idea is to make these ACL linkages with identical solid circles such as wire, or rods, or flat material. The circles can be made of different shaped polygons for illustrative purposes. The mathematical properties of ACL's should be amenable since they are a much simplified version of topological linkages. This note will show that ACL's have some interesting properties of their own. Some of the properties of ACL's remain valid if the solid circles are changed to topological strings thus results might have some topological use.

There are several ways to designate an ACL some of which shall now be listed. The most obvious is to use knot theory ideas and write the number of circles and the linking method and the number in a list that the specific ACL belongs to. Another method is to list the main variables that a group of ACL's with n circles could satisfy. You can also writ e a specific exact construction code formula for an ACL.

Two main linking methods will be used. The first refers to the number of circles not linked thru by each circle. A regular ACL of n circles means each circle links through the identical number of circles in the linkage. Two constants can be used, k and q where k+q=n. The constant k means the number of circles not linked through while q means the number of circles linked through. For n links if every link links once through every other link then k=1 since it does not link through itself. Then k is a constant for any n, and q is variable so we only need use k in that ACL designation. If every link links through a constant number q of n circles then k is variable so we only need use q in that ACL designation. When k is 1 the system is toroidal. In topology a great deal is known about this type of system but not as much about solid circles able to be rearranged. When q is 2 or greater the system forms chains of circles connected in a loop. This gives the first two variables to be used by writing ACLk1 when k=1 or kx where x is a positive integer greater than 0 or ACLq2 when q=2 or qx where q is a positive integer greater than 1. We shall mostly consider the case for k1.

The system can be extended to include mixed groups of ACL's linked together where ACL's in the group can have their own k or q designation thus several ACL's with their k's or q's could be listed for one overall ACL linked system of solid circles. There would need to be a way to show how the different ACL's are linked together. These systems are not regular ACL's and will not be considered here. Complicated systems like this might be investigated with a computer.

The next variable in an ACL group is u or the number of unit circles that cannot be made close to any other circle in the ACL. Thus ACLk1,u. A close braid consists to two or more circles that are all linked positively or negatively as a single group. Two linked circles can lay over each other in two ways to touch closely around their circumference. One way gives one positive twist and the other way gives one negative twist.

Figure one shows how the same two linked circles can be seen to twist one way or the other. If another circle is linked to the first two then the twist produced by how the circles are laid over is locked in. This is a close braid of three links. Close braids of any number of links are possible. This variable is called m and refers to the number of close braids in an ACL, so we have ACLk1,u,m. The next variable is the number of circles in the linkage, called n and we have ACLk1,u,m,n. Another variable can be used called p and refers to parallel links, where a link can be moved to sit next to another link but does not link through it. This variable can refer to cases where k is 2 or greater and gives the number of groups of parallel links. For instance take a k1 linkage and add a parallel link beside every link. Then k1 becomes k2, n increases 2*(old n), and p equals (new n)/2=old n. Note that k could also be greater than one without any parallel links in the ACL.

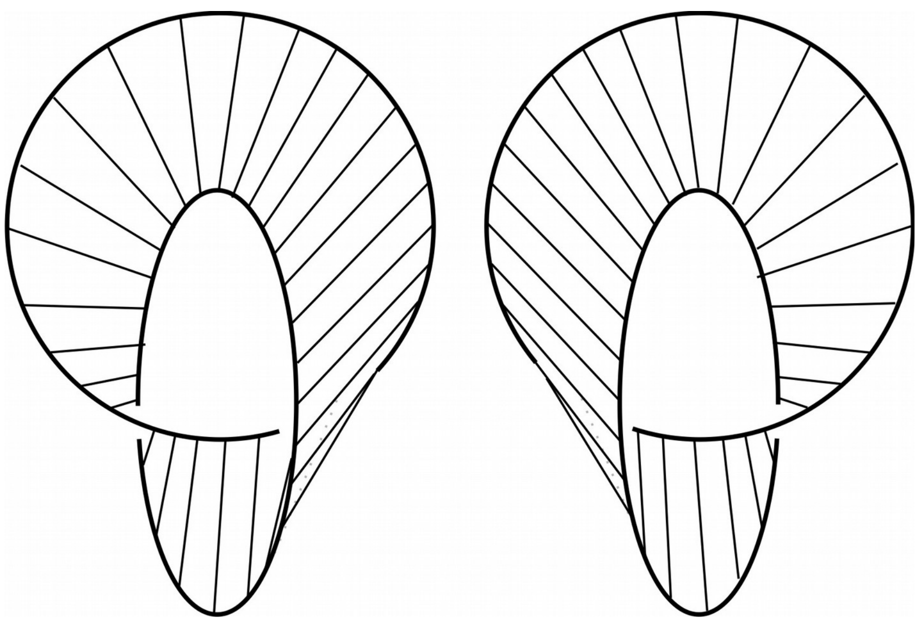


Figure 1

Figure two shows the way circles can lay over and some of the ideas discussed so far.

ACLk1,0,1,2=+2 ACLk1,0,1,2=-2 ACLk1,0,1,3= -3 ACLk1,0,2,5= -2,+3 ACLk1,0,1,14=+14

Figure 2

Figure three shows an m=6 ACL that has been linked by alternating the twists at all levels such as for close braids, close close braids, etc. It has the special property of having a circularly symmetrical toroidal display where the inner part is compacted and the outer part is dominant. It is called a CLA for close link alternating. By manipulation the inner part can be brought out reversing the order of dominance. This system can be extended without limit both for number of close braids and number of circles each close braid is composed of. The text with Figure three shows how each close braid can be either reduced in steps to unit circles or built up from unit circles. By using this method of building or reduction we have an exact method of building a simple close braid ACL. Close reduction: If all six close braids in the last formula are reduced to a single circle then you have the 4th formula. Reduce again to get 2 close braids in the 3rd formula. Reduce again to one close braid of the 2nd formula. A final reduction gives a unit circle, the 1st formula. With an ACL that reduces to a prime this removes the close braids, and close close braids, etc. but leaves the prime number of unit circles at the final reduction instead of a single unit circle. However a prime enlarged by close braids cannot form this kind of display.

Figure 3

ACLk1,1,0,1={o} o=orange start with a single orange circle
ACLk1,0,1,2={o,p} p=purple link purple to orange
ACLk1,0,2,4={+(o,b),-(g,p)} b=black, g=green Lay ob+, link gp- to it.
ACLk1,2,2,6={+(-(o,w),b),-(g,+(r,p)} w=white, r=red Link w to o as -, link r to p as +.
ACLk1,0,6,24={+(-(+4.o,-4.w),+4.b)-(-4.g,+(+4.r,-4.p))} Add the close links per signs

The formulas give the assembly instructions starting with the first formula.
Moving downward and adding links and close braids at each step builds up this ACL.
Moving upward and reducing the close braids at each step reduces this ACL to one circle.
The signs applied to make the 4 circle close braids simply alternate on the 4's from left to right.
This ACL (bottom formula) has six close braids. To get this kind of symmetrical toroid

display the signs must fully alternate. Left is a linear arrangement that simulates the assembly formula. In the middle figure the purple, 4.p is the inner toroid, then red, 4.r, 4.g, 4.b, 4.w, 4.o. The rightmost display is in the opposite order of dominance. Only these two fully toroid displays with circular symmetry are possible. This kind of alternating structure can be made with any number of close braids thus making it more and more complex but yet quite simple. To add more close braids would require a smaller d/D or larger circles. This structure of 24 circles is already very tight when symmetrically displayed.

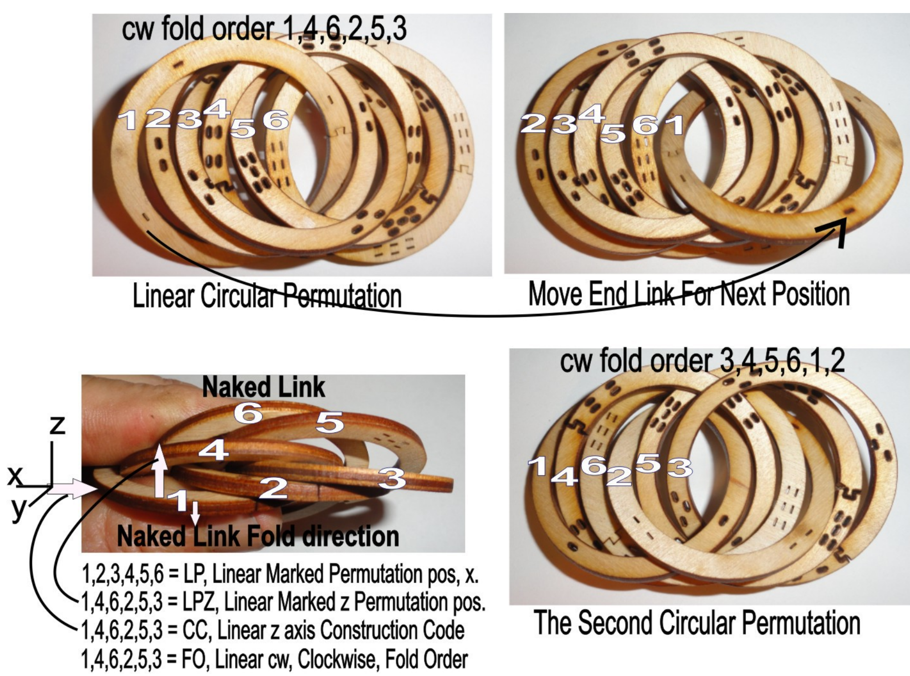
Figure 4 shows an m2 ACL made with heavy gauge copper wire. You can easily display it with 3 circles dominant or four circles dominant or as a linear arrangement, left. The circles can be thickened to get a tighter display like Figure 4. A very approximate formula gives n=(2D/d)-2 for a fairly tight symmetrical display. If you want a system that lays more flat on a table top make d/D fairly small. You can also make flat circles by cutting out of sheet material so they lay more flat making it easier to see how they arrange circularly and linearly and other ways.

Figure 4

ACLk1 Primes:
The prime ACL's have their own designation. The first and smallest possible prime is ACLk1,5p. It is made from 5 circles. A prime ACL occurs when no circle in the linkage can be made close to any other circle in the linkage. The five prime is its own mirror image and only one 5p exists. Figure 5 shows 5p with individualized (marked) circles that can be solved for five different symmetrical displays.

Figure 5

An ACL n prime can be arranged linearly with only two different circular linear permutations or 2n ways(see below). Six prime can easily be made from 5p. Put 5p in a linear arrangement then fold or lay over the circles so that no foldable end link, NFEL, occurs on either end of 5p. A foldable link, FL, is a link that can be folded, or laid over about the horizontal axis through the center of the linear display. If an end link can be folded it is a Foldable End link, FEL. By making both end links not foldable, NFEL you are set up to create a p+1 prime. Link the 6th circle through the center of the 5p and you have 6p. Since all the ways to eliminate foldable end links, FEL from 5p give the same basic fold arrangement only one 6p is possible using this method. For 7p the situation becomes more challenging. Six prime has 2 different arrangements from each of the 2 possible circular permutations with NFEL on either end and several with different looking fold arrangements. This means more than one 7 prime must exist. At present the exact number is not known but it is probably fairly small. Folding one at a time two different circles are available to fold over in a linear prime, LP arrangement. These represent the two directions of folding about the horizontal axis. Folding in one direction n times for an n circle prime completes one fold cycle in one direction. So folding in either direction gives exactly the same set of n fold arrangements. Of these n-4 produce ends with NFEL. Thus 5p has one no foldable end link position for each linear permutation that can be used to build 6p(but all are equivalent). Then 6p has two NFEL fold positions per n circular permutations and so on. Since 2n permutation arrangements are possible we have a maximum possibility of 2n(n-4) different ways or 42 ways to make 7 prime. Many of these will be found to be geometrically equivalent.



6 Prime(made from 5 Prime)

Figure 6

Figure 6 shows two different linear arrangements for 6 prime. Figure 6 also shows how the fold order looks from a top view or y axis. The main problem of the primes is to find a formula for the number of primes for n prime if such a formula exists. The problem of understanding the two LP or linear permutations is mostly solved as will be discussed.

Figure 7

Figure 7 shows a 7 prime in a symmetrical display. This can be seen by rotating the photo 180 degrees and looking for differences. If none can be found it is rotationally symmetrical in this display position. It is unknown if it has other symmetrical displays. Many larger primes will have symmetrical displays. These can be discovered with the matrix methods discussed below. Other ways to make primes exist but have not been investigated. For instance add a link to 5p to produce a close link that can then be folded oppositely to its close group. As long as this is not an end link adding a link to this arrangement should produce a 7p. How fast do the number of different kinds of n primes increase as n increases? Is this rate of increase smooth or is it choppy?

Linear Permutation Matrices `

As discussed any prime ACLk1 has very restrictive rearrangement properties so that only 2n circular linear permutation arrangements are possible for a given construction and linear marking order. However each of these can be written in reverse order making four initial LP arrangements each beginning on the left with the circle marked 1. All four are necessary to show how 2 linear permutations exist for every prime.
Construct a 5 prime in linear marked order as 1,2,3,4,5. It has four LP or linear permutation matrices, 4 CC or construction code matrices and 4 FO or fold order matrices. The first two CC and FO matrices are derived from the first row of each LP as shown in the lower left of Figure 6 by folding each next foldable link clockwise and then writing its CC for that row. The FO row for that CC row can then be written. Each CC row number is just the y axis position of each circle moving left to right. FO is the

order of clockwise folding each link of the CC row moving left to right. A third matrix is LPZ derived by recording the marks along the z axis from front to back. We want to begin each CC matrix with a circle marked 1 so that 1,2,..,n simplest marking and its reverse 1,n,n-1,..,2 are considered.
LPZ always comes in four varieties as do the CC and FO matrices and LP is not a matrix. One of the four LPZ will match the first row of a CC matrix based on a start marking of 1 to n and linear arrangements always starting left to right with a 1 mark.
5p LP1a= 1,2,3,4,5

5p CC1a=	twist	5p FO1a=	LPZ1a=	5p LP1a= 1,2,3,4,5
1,3,5,2,4	+4	1,4,2,5,3	1,4,2,5,3	
5,2,4,1,3	-4	4,2,5,3,1	4,2,5,3,1	
4,1,3,5,2	0	2,5,3,1,4	2,5,3,1,4	
3,5,2,4,1	-4	5,3,1,4,2	etc.	
2,4,1,3,5	+4	3,1,4,2,5		

5p CC2a=		5p FO2a=	LPZ2a=	5p LP2a=1,4,2,5,3
1,4,2,5,3	+4	1,3,5,2,4	1,2,3,4,5	
5,3,1,4,2	-4	3,5,2,4,1	2,3,4,5,2	
4,2,5,3,1	-4	5,2,4,1,3	3,4,5,2,1	
3,1,4,2,5	+4	2,4,1,3,5	etc.	
2,5,3,1,4	0	4,1,3,5,2		

5 p LP1b=1,5,4,3,2

5p CC1b=	5p FO1b=	LPZ1b=
1,3,5,2,4	1,4,2,5,3	1,3,5,2,4

LP2b=1,3,5,2,4

5p CC2b=	5p FO2b=	LPZ2b=
1,4,2,5,3	1,3,5,2,4	1,5,4,3,2

The CC and FO matrices are two dimensional permutation matrices. The 5p is its own mirror image. To generate the 5p CC matrix numerically when the first row is listed change 5 to 1 and subtract 1 from the others for the next row CC. Note CC and FO does not depend on individual markings on the circles.
For FO fold order matrix just use count order of CC matrix, left to right by noting each next position that could be folded cw, the x position of 1 then the x position of 2, etc. giving 1,4,2,5,3 for 1st row of the 5p FO1 matrix. The following rows are then just a circular permutation of each previous row.
With a simple rule you can create a prime CC listing.
Non Close Rule, NCR: any two adjacent CC numbers must have absolute difference greater than 1 for each number in the circular permutation. Using this rule you can write a set of numbers, 1 thru n to produce a prime of any size n. You can then generate the four CC matrices and by comparing to any similar size CC matrix determine if the prime is new. If any two rows repeat between old and new matrix the prime cannot be new.
The NCR leads to the idea of a difference sum. Subtract all adjacent numbers in a CC and calculate the

resulting absolute difference sum, dsum. A larger dsum means a larger neighborly separation of the circles.

Here are the matrices for a 6 prime built from 5 prime in Figure 6.
For linear marked circular permutation LP1a=1,2,3,4,5,6 we have:

6p CC1a=	dsum	twist	6p FO1a=	LPZ1a=	LP1a=1,2,3,4,5,6
1,4,6,2,5,3	16	+3	1,4,6,2,5,3	1,4,6,2,5,3	
6,3,5,1,4,2	18	-7	4,6,2,5,3,1	4,6,2,5,3,1	
5,2,4,6,3,1	16	-5	6,2,5,3,1,4	6,2,5,3,1,4	
4,1,3,5,2,6	16	+5	2,5,3,1,4,6	2,5,3,1,4,6	
3,6,2,4,1,5	18	-1	5,3,1,4,6,2	etc.	
2,5,1,3,6,4	16	+5	3,1,4,6,2,5		
	total	0			

For the second linear marked circular permutation LP2a= 1,4,6,2,5,3 we have:

6p CC2a=	dsum		6p FO2a=	LPZ2a=	LP2a= 1,4,6,2,5,3
1,4,6,2,5,3	16	+3	1,4,6,2,5,3	1,2,3,4,5,6	
6,3,5,1,4,2	18	-7	4,6,2,5,3,1	2,3,4,5,6,1	
5,2,4,6,3,1	16	-5	6,2,5,3,1,4	3,4,5,6,1,2	
4,1,3,5,2,6	16	+5	2,5,3,1,4,6	etc.	
3,6,2,4,1,5	18	-1	5,3,1,4,6,2		
2,5,1,3,6,4	16	+5	3,1,4,6,2,5		
	total	0			

By permuting LP1a to LP1b= 1,6,5,4,3,2 we get have:

6p CC1b=	6p FO1b=	LPZ1b=	LP1b= 1,6,5,4,3,2
1,5,3,6,2,4	1,5,3,6,2,4	1,3,5,2,6,4	

By permuting LP2a to LP2b= 1,3,5,2,6,4 we get a have:

6p CC2b=	6p FO2b=	LPZ2b=	LP2b= 1,3,5,2,6,4
1,5,3,6,2,4	1,5,3,6,2,4	1,6,5,4,3,2	

Determining twist for primes or any ACL is always based on a specific lay over position and is not an invariant. Total twist in any matrix CC row can vary because the circles are laid over each other in different ways in each row. This CC matrix can be done with a single close braid as well. For any given row of a CC matrix a twist can be calculated for that specific lay over of circles. Calculate the twist of each row of the CC by starting at the left. If the second number is greater than the first it adds +1 if less it adds -1. Compare the third number to the second and first number and add or subtract a one for each if greater or less. Continue with the fourth number compared to the third, second and first and so forth with each number the number near the last number. The twist calculated for a given row is the same if the linear array is flipped over left to right and renumbered for the CC or flipped 180 degrees about its horizontal axis which does not change the CC numbering. This twist has a min-max of $\pm((1/2n^2)-(1/2n))$ for close braids but is smaller for the prime linkages. For instance reversing the first row 1,4,6,2,5,3 in the 6p CC matrix above renumbers as 4,2,5,1,3,6 and each of these gives a twist of +3. If the numbers of a CC row are listed in reverse order a mirror image linkage results and the sign of the calculated twist reverses. This twist is based on the accepted twist for two linked strings in a planar projection as can be found in reference number 3.

Here are the CC matrix for seven prime.

LP1a=1,2,3,4,5,6,7		LPZ1a= 1,3,5,7,2,6,4
7p CC1a=	twist	7p FO1a=
1,5,2,7,3,6,4	+7	1,3,5,7,2,6,4
7,4,1,6,2,5,3	-5	3.5.7.2.6.4.1
6,3,7,5,1,4,2	-9	5,7,2,6,4,1,3
5,2,6,4,7,3,1	-5	7,2,6,4,1,3,5
4,1,5,3,6,2,7	+7	2,6,4,1,3,5,7
3,7,4,2,5,1,6	-1	6,4,1,3,5,7,2
2,6,3,1,4,7,5	+7	4,1,3,5,7,2,6
total	+1	

LP2a=1,3,5,7,2,6,4		LPZ2a= 1,2,3,4,5,6,7
7p CC2a=	twist	7p FO2a=
1,3,5,7,2,6,4	+7	1,5,2,7,3,6,4
7,2,4,6,1,5,3	-5	5,2,7,3,6,4,1
6,1,3,5,7,4,2	-1	2.7.3.6.4.1.5
5,7,2,4,6,3,1	-9	7,3,6,4,1,5,2
4,6,1,3,5,2,7	+3	3,6,4,1,5,2,7
3,5,7,2,4,1,6	-1	6,4,1,5,2,7,3
2,4,6,1,3,7,5	+7	4,1,5,2,7,3,6
total	+1	

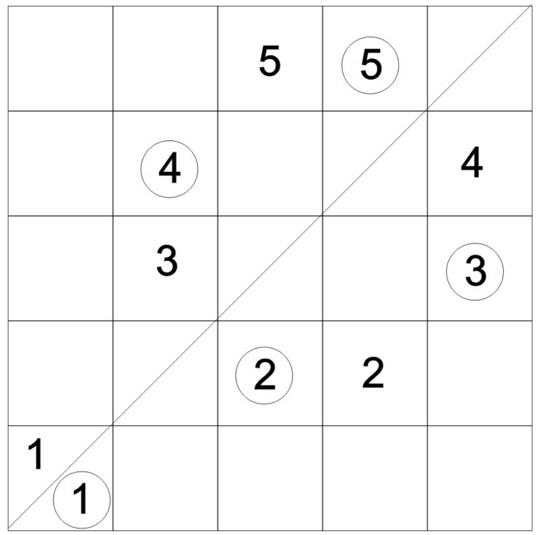
7p

LP1b= 1,7,6,5,4,3,2	LPZ1b= 1,4,6,2,7,5,3	LP2b=1,4,6,2,7,5,3	LPZ2b= 1,7,6,5,4,3,2
7p CC1b=	7p FO1b=	7p CC2b=	7p FO2b=
1,5,3,6,2,7,4	1,5,3,7,2,4,6	1,5,3,7,2,4,6	1,5,3,6,2,7,4

Notice that for all 3 example prime matrix we have the following results, discussed again in the numbered properties below.

The following relations hold for the first row of each matrix:
The CC to FO also holds if you start a LP with a marked circle other than 1 if the CC starts with a 1.
FO(CCx)=FOx, FO(FOx)=CCx
LP1a=LPZ2a LP1b=LPZ2b LP2a=LPZ1a LP2b=LPZ1b
CC2b=FO1b CC1b=FO2b CC1a=FO2a CC2a=FO1a=LP2a
LP1b= 1,(reverse(LP1a 2 thru n)
LP2b= 1,(reverse(LP2a 2 thru n)

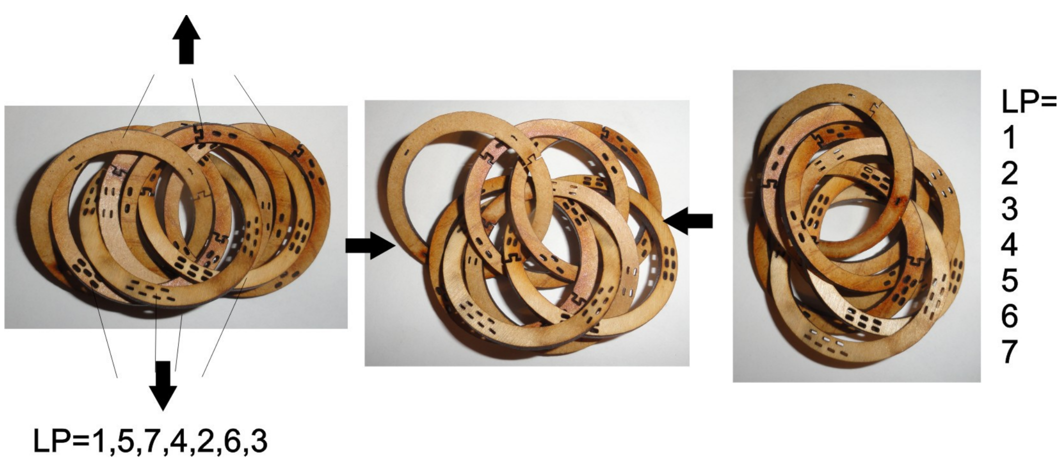
Of course the LP1a must be 1,2,...n.
This works even though each matrix is derived by a different method of tabulation showing a strong entanglement of the ACLk1 prime circles. If you plot CC1a and FO1a along the x,y axes you get a symmetry of CC to FO about the 45 degree diagonal passing through the origin to the right. This has been done for 5 prime in Figure 8. Since CC1a = FO2a then CC2a must = FO1a. Knowing LP1a and CC1a all the other codes can be easily determined.



5 Prime CC1a=1,3,5,2,4=FO2a
CC2a=1,4,2,5,3=FO1a

Plot of CC1a and CC2a(circles) showing symmetry across the main diagonal.

Figure 8



A simple physical method for finding the other basic LP.

Figure 9

Another way to derive the other LP is shown in Figure 9. Grasp all the lower circles leaning upward and all the upper circles leaning upward with your other hand. Pull apart carefully along the y axis. Now push in along the x axis until a new linear arrangement is achieved. You can see partially or fully what the new LP will be before starting by noting the level order of the circles. This shows why FO1a always equals CC2a and FO2a=CC1a, etc.

Interesting properties of the ACLk1 prime CC, FO and LPZ matrices:

1.0 Non Close Rule, NCR: any two adjacent CC numbers must have absolute difference greater than 1 for each number in the circular permutation. Using this rule you can write a set of numbers, 1 thru n to produce a prime row CC of any size n. This rule does not assure a prime(see 5.1)

2.0 Generating an FO row from a CC row generates the FO for that row thus FO(CC1)=FO1. Generating an FO row from a FO row generates the CC for that row thus FO(FO1)=CC1

3.0 Flipping the linkage over left to right and rewriting the CC gives the same twist calculation as the non flipped CC. This provides a crosscheck of the twist calculation procedure.

4.0 An FO row can be treated as if it were itself a linkage, it obeys the NCR rule. Generating a CC matrix from the first row of FO1 produces CC2,1, or CC(FO1,1)=CC2,1. Generating a CC matrix from the first row of FO2 produces CC1,1, or CC(FO2,1)=CC1,1

5.0 Calculating twist of an FO row results in the same twist as for the CC row that generated it.

6.0 Rewriting a CC row by reversing the numbers, left to right produces the mirror image CC linkage with the same twist value but with the sign reversed.

7.0 The 5 prime linkage is its own mirror image and its total matrix twists add to zero. The CC1 and FO1 for 5 prime are reverse permutations of each other as are CC2 and FO2. This works because 5 prime is its own mirror image. If this property that holds for larger symmetrical primes it would be useful to find mirror symmetry primes.

8.0 LPZ records the order of marks from front to back of the linear arranged prime while LP records marks from left to right or x axis. LPZ is therefore a psuedo rotation of LP 90 degrees about y. Knowing these limitations on the four matrix, LP, LPZ, CC, is a preliminary proof that ACLk1 symmetrical primes have exactly two unique permutations. In fact the NCR seems to require only one LP but the symmetry of CC to FO and permutation property indicates the primes will always generate the two LP as demonstrated above.

Properties and rules of the ACLk1 primes:

1.1 An ACLk1 prime can have no close braids or links.

2.1 The smallest prime is 5p and is its own mirror image.

3.1 A foldable link, FL in a prime is defined as a link in a flat, left, right linear arrangement that can be folded toward the user clockwise down, or anticlockwise up. Every linear arrangement always has two oppositely fold-able links available one of which can be chosen to fold.

4.1 An n prime can be used to make an n+1 prime. Fold a linear flat arrangement so that each end has no foldable end links NFEL, then add a new link to make the n+1 prime

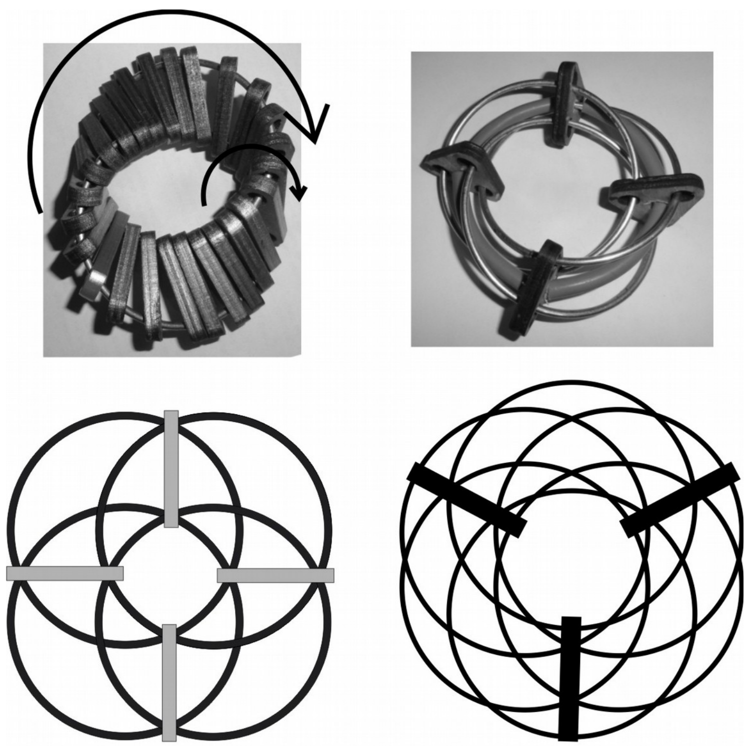
5.1 Any two primes can be added to make a close prime braid, CPB. The resulting structure is not a prime. It lacks some of the properties of a prime such as method of folding and permuting. Think of each prime as a one circle with no foldable end links NFEL. Now you can Link them to make their sum p1+p2. The result is a kind of prime close braid where m=2(p). More primes can be added this way to obtain CPB with a complex close twist of primes such as +3(p), etc.
7.1 The twist of a close braid can be calculated by its fold matrix the same as for a prime. Since close braids all have a simple sequential structure this twist can be given as function of n, t=n^2-n-

2. A close braid with marks can be permuted in 2n(n!) ways if marks are considered but each CC is Multiplied by 2 since a close braid can twist two ways. Since the links of a close braid can be placed in any order it is practical to mark each the same such as 1 or red.

8.1 A prime of primes can be formed by changing each circle in a prime to a prime. This combined structure will not have the properties of a prime but once again you can think of each prime as a

single circle and permute the individual primes like a single prime.
9.1 A prime in a linear arrangement always has n-4 positions that have NFEL's.
10.1 Every prime has two unique LP's, or marked linear circular permutations.
11.1 The number of possible n primes ACLk1 {np} is smaller than the total number of possible ACLk1 {n}. It is an open question how {np} relates to {n}.

Nodal display devices using ACLk1 links:
The special properties for ACLk1 will not all hold if any one link links thru some but not all the other links. This would mean that k does not equal one for that ACL linkage. Continuous double axis toroidal motion for all links will not be possible. Fold order for primes will not not work per rules. The circular permutation for primes will have flaws. Full toroidal systems with their interesting properties are only possible if k=1, every link must link once thru every other link. When k=1 the double axis toroidal motion is always possible no matter how many circles there are or how they are linked by laying over in different ways as long as they are thin enough.
Two steel circles can be linked by a series of of bars of equal length with a hole for a circle to pass thru at each end. This linkage can then be make to rotate by rotating the bars continuously about the toroid axis. When the bars rotate thru 360 degrees the two circles orbit around each other by 360 degrees. Thus the bars rotate about the toroid axis while the circles rotate in a double axis(toroidal) manner about each other. The circles never turn upside down, instead traversing around the main axis of the virtual torus. This can be done with more circles as shown at top right in Figure 10.



Nodal controlled toroidal motion displays

Figure 10

The two bottom diagrams in Figure 10 show how you can trap nodes in a symmetrical manner to produce a toroidal motion device that passes from a flat shape to a cylindrical shape in a continuous manner. The circles do not turn over. only the bars do. Two devices are shown as one with four circles and one with six circles. These devices are called Tetraka(also Abraka for the 4 circles) and Hexaka. This could be done with any number of two or more circles if they have a small enough d/D. A Diaka, Triaka and Pentaka have also been made. If the bars were in the shape of flat circles with two opposed holes they would coincide with the approximate surface of the enclosing virtual torus. The toroidal radius of this torus is equal to the radius of the circles. When the bars turn over each circle must move to occupy the same bar holes. Since each circle occupies a different set of holes the motion might seem complex but it is straight forward. Is it possible to make an ACLk1 device where a Pentaka flexed to a cylindrical shape could intersect another Pentaka in a flat shape with the result being also double axis rotatable? What is the mathematical equation of continuous motion of each circle? What is the formula for d/D for producing a workable xAka?

The most interesting thing about the ACLk1 system is the variable twist properties of a linkage. In a sense a linkage can have several values of twist almost simultaneously. Depending on your point of view a two circle linkage can have +1, or -1 twist.

Figuring out which linkages are actually different from each other for a given n becomes more of a problem as n increases. This use of CC matrices shows that the system is quite precise and displays a unique ordered entanglement. ACL's are worth investigating, not least because the circle is considered to be one of the simplest possible forms.

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