Integer Code For Linked Circles

Presenting some new integer invariants

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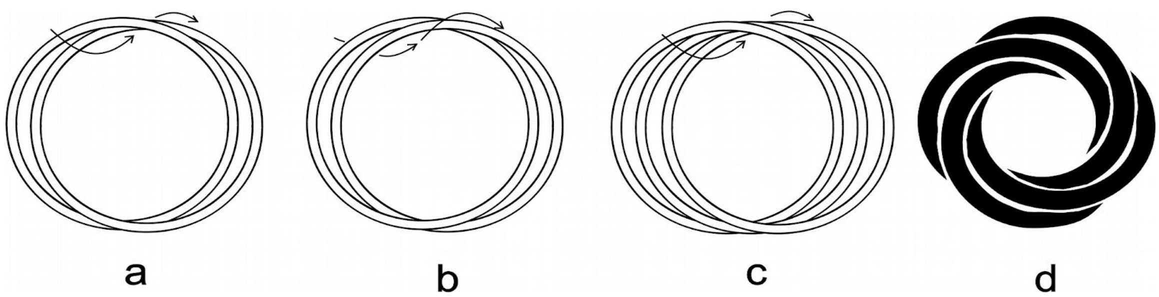
By Douglas A. Engel

If c_n is the number of distinctly different ways of linking n identical circles so that every pair is linked, then as n increases c_n becomes more difficult to determine. If the circles can be arranged so that their centers coincide with a line the linkage is linear and there is a simple integer code CC for constructing it. This code will also allow operations such as linear rearrangements and twist calculations for any linear arrangement. When you get to a linkage of seven or more circles it is possible to make the linkage in such a way that a linear arrangement is not possible. This is called a **Rogue link**. To add insult to injury it turns out that there are exponentially more Rogue links than there are linear links as n increases. Restricting links to be circles would seem to make this a simple exercise but the different kinds of links possible actually follow a curve of increasing complexity as n increases. In addition the twist increases dramatically for each added link when n grows large. An interesting bonus is that the regularity of using identical circles produces a hierarchy of different kinds of structures as n increases that make the system mathematically interesting

Define '**All circle links**' $\mathbf{A_{CL}}${} as the set of equal sized linked torus(circle) shapes. Each torus has major and minor radii **R** and **r** and major and minor diameters, **D** and **d**. We can make the linked circles thinner as needed until d/D becomes 0 in the limit, equivalent to a geometric circle. If d/D>0 only a certain maximum, n, of these circles can be linked so that every circle links thru every other circle. For small values of n this maximum has an approximate value of n=(2D/d)-2, so n can be made larger by making d smaller or D larger. This is because as n increases there is less and less room for n-1 circles to pass thru a single circle due to their thickness, or small diameter d.

Given n links, each linking to n-k=q links , then if every link links once through every other link then then it does not link through itself, so k=1, and q=n-1. Since q varies with n but k is constant we have the set description $\mathbf{A_{CL=}}${k1}. The Hopf link consists of two linked circles where each circles d center passes through the D center of the other circle at right angles. The two circle link is the simplest nontrivial link[5]. $\mathbf{A_{CL=}}${k1} circles need not pass through each others centers so Hopf links are only mentioned here. For linear $\mathbf{A_{CL=}}${k1} and if the circles are thin enough you can lay them out on a surface in a projected **flat linear array, L**, defined as having the center of each circle lie on a straight line. Thus we add L to the set of $\mathbf{A_{CL=}}${k1,L} links we will be concerned with.

A simple integer construction code CC describes the exact construction of any $\mathbf{A_{CL=}}${k1,L}. Define a **close link** as a linkage of two or more circles that are linked so that each circle twists in the same direction around the other circles. We can use twist in place of linking number because circles cannot bend like a ribbon so writhe = 0. This makes twist calculations here similar to linking number calculations as defined in knot theory[1]. Each pair of circles in the close link can be manipulated and made to touch around their circular torus surface. We need this definition because as will be seen below links of five or more circles can be made where no close links are possible, called a **prime link**. Thus 5 prime, 5p is the smallest possible prime.



$A_{CL=}${k1,u0,m1,n2,L}, twist = +1 **twist = -1** **$A_{CL=}${k1,u0,m1,n3,L}, twist = 3**

Figure 1

Figure 1, a and b shows two linked circles laid over or projected in two ways as close links. The pair on the left have a construction code of CC=1,2 while the pair on the right have CC=2,1. Links of a close link can be seen to twist in the same direction as shown in Figure 1. The smallest close link is two linked circles. The two circles in Figure 1, a have one positive twist while the two circles of the identical linkage in Figure 1, b have one negative twist [1]. If more circles are added to a close link the twist becomes more 'locked in' but can still vary by laying the circles over in different ways. Close links of n links allow the n circles to be placed in any marked order or n factorial possible ways. The total twist is maximum when the entire link of n circles is a single close link in a close projection. A simple way of calculating the twist for any linear flat projection **$A_{CL=}$**{k1,L} link will follow below. For a linkage of n circles **u** is the total number of circles that cannot be made close to any other circle.

Close links of different twists can be linked together in various ways. This variable is called **m** and is the number of different close links. For instance you could have a link where 5 close links twist one way linked to another where 3 close links twist the opposite way making m=2, u=0. It may seem that u and m are redundant used together, but for many links they are not redundant. It is evident that if m=0 then u=n. This is called a prime link, mentioned above, where no pair of links are close(cannot be made to come close by manipulation of the circles). We also have the total twist of the flat projection, 0, +t or -t. All of these variables, including twist, will not guarantee a unique linkage. This becomes more prominent as n increases. By folding each circle of **$A_{CL=}$**{k1,L} over for a cycle of n folds n different layover arrangements occur completing the cycle. This produces n rows of n codes per row which is a CC matrix. The twist for each linear fold arrangement is summed per row then the row twists are summed for a matrix total twist M_{TT}. The M_{TT} is always a constant or invariant for that linkage for all possible unique CC coded folding matrices.

Figure 2 top left and right shows a **linear 5 prime linkage** of five circles. The bottom left shows the result of one fold of the top left CC while the bottom right shows the fold order this creates. Five prime is the smallest possible prime so that u=n=5. Refer to the top left part of Figure 2 to see how a **construction code**, **CC**, is written for this linkage. The leftmost circle is in the level 1 z axis position. The next to leftmost circle is in the level 3 z axis position and continuing we can write the level, or z axis position of each circle moving from left to right along the x axis as 1,3,5,2,4. This will be the first

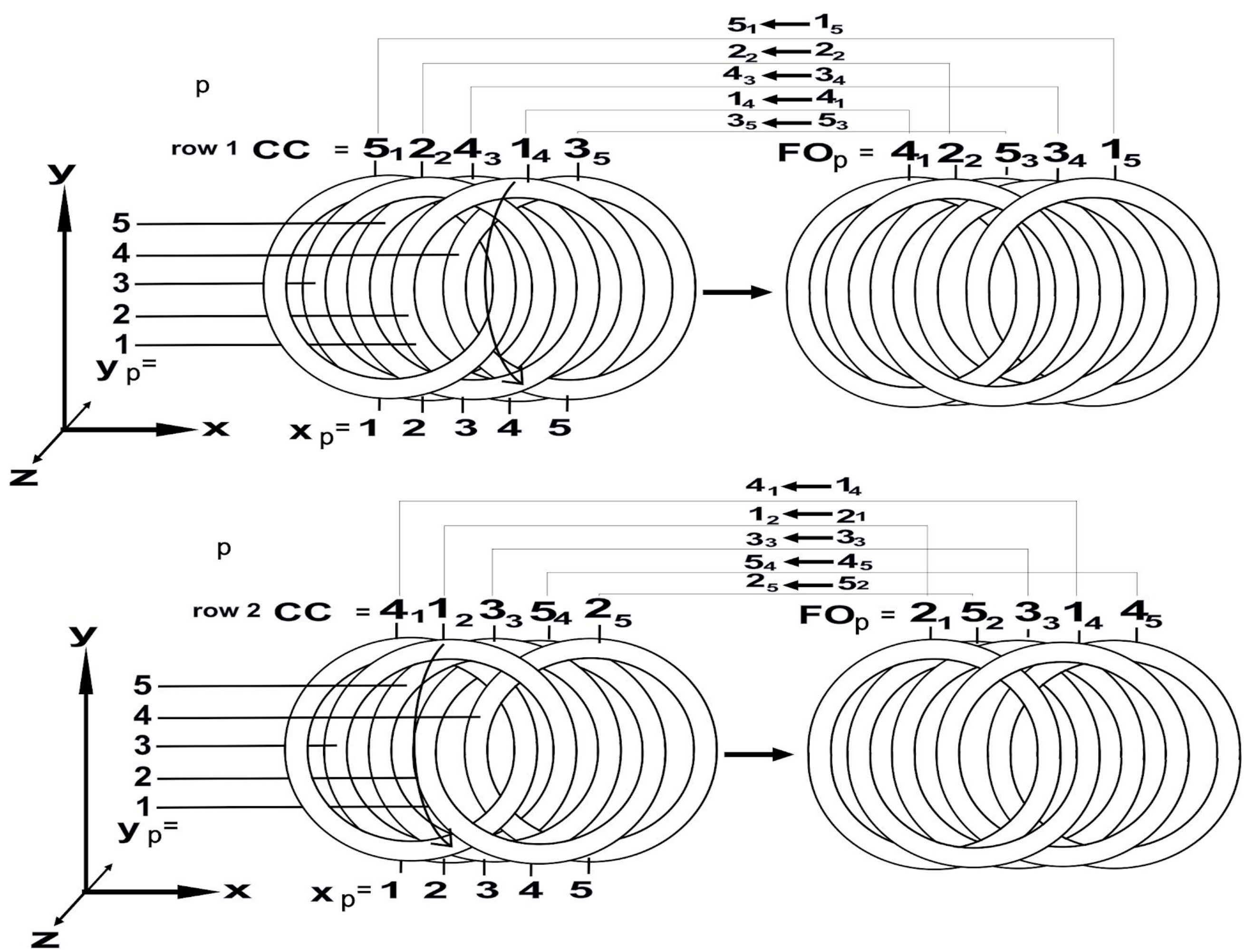


Figure 2

row of a **CC matrix**. It is a convention to put the left circle in the level 1 position when writing the first row of a CC matrix. Level one is always the first circle that can be folded downward for a new layover position as shown by the curved arrow on the left group of circles. These matrices will be discussed below.

The **fold order**, or **FO** is written based on going along the x axis and writing the x position of the first circle in z level 1, then write the next x position of the circle in z level 2 and so on. This is the order of folding of the circles when folding each circle one at a time down or toward you. First fold down(about the x axis) the circle in position one in the listed fold order then the next circle able to fold is in the listed position 4 and so on. A prime requires that u=n so that no pair of circles can be made close to each other. For a prime any two adjacent integers in the construction code or CC must have a difference greater than 1. You can pull the circles apart in two groups oppositely along the y axis to put the linkage into a position where CC and FO exchange places. Pull 1,4,2 toward you and 5,3 backward. Figure 4 shows this process for a 6 prime linkage. Because no folding needs to take place to do this the **twist of CC always equals the twist of its FO**, and CC and FO can exchange places enabling the full complement of all 2n folding matrices.

The twist of a linear **$A_{CL=}$**{k1,L} can easily be calculated from the CC. Twist can vary depending on how the circles lay over each other in pairs. The twist for a given CC row then is the sum of the twists of each ordered pair of circles. Because the linkage is linear twist can be calculated by looking at all possible left-right ordered pairs of numbers in the CC or the FO thus:

twist(CC,L) = sum [(CCp_iCCp_j) of all pairs $i<j$ and $CCp_i<CCp_j$: t = +1 and $CCp^i>CCp_j$: t = -1]
twist(FO,L) = sum [(FOp_iFOp_j) of all pairs $i<j$ and $FOp_i<FOp_j$: t = +1 and $FOp_i>FOp_j$: t = -1]

For instance the numbers a,b form exactly one ordered pair, a,b, while a,b,c produce 3 ordered pairs a,b, and a,c, b,c. Thus a linear CC of n numbers produces n^2/2 – n/2 ordered pairs. Each ordered pair has twist of +1 if $a<b$ and twist of -1 if $a>b$. The calculation for 5 prime CC=1,3,5,2,4 then results in t(1,3,5,2,4) = 4 The calculation for the 5 CC rows of the matrix formed by doing one fold for each new row results in a total matrix twist of zero. Five prime is its own mirror image.

The twist calculation for a linear $A_{CL=}${k1,L} works for any set of positive sequential numbers 1 thru n.
Given CC=1,3,5,2,4 then fold order, FO =1,4,2,5,3 as shown in Figure 2. This can be described mathematically by exchanging the position p and CC numbers so that CCp becomes pCC then re-position so that CC is in position order p(1,2,3...n) and p is then the FO.

CCp >> pCC >> Fop(pCC)=FOp >>FO

thus CCp = 1_1, 3_2, $5_{3,}$ 2_4, 4_5
pCC = 1_1, 2_3, 3_5, 4_2, 5_4
FOp(pCC) = FOp = 1_1, 4_2, 2_3, 5_4, 3_5,

Note that for certain number sequences CC sometimes exactly equals FO. This is the case for CC=1,4,6,2,5,3 = FO=1,4,6,2,5,3. For the nxn code matrix row-1 twist of CC equals row-i twist of its FO. CC and FO are complementary operators where CC(FO) = CC and FO(CC) = FO.
The code allows any simple list of sequential integers to be rewritten in 1 thru n form(ex 6,4,5 would be 3,1,2) to analyze the number pattern in comparison to other patterns along with their twist numbers.

Some Rules for a linear $A_{CL=}${k1,L} to be prime:
1. No close links are allowed so that code numbers a_i satisfy $n-1>|a_i-a_{i+1}|> 1$ and $n-1> |a_1-a_n|> 1$.
2. **$A_{CL=}$**{k1,L} is not a composite prime, px#px.

A method for building an n+1 prime from an n prime:

Put the n prime linkage in a position where left and right code numbers are not equal to 1 or n:
$1<CC_1<n$. $1<CC_n<n$.
This brackets the two ends with circles that are not ready to be folded. Now add a link to one end. Five prime has only one position like this available to make 6 prime. Thus only one 6 prime is possible, 6 prime and its mirror image.

Linear Permutation Matrices `
An nxn matrix is our goal. To generate the next matrix row of a CC of n links by using mod the process is:

CC-1 mod n
example if row 1 = 1,5,2,7,3,6,4
producing row 2 = 7,4,1,6,2,5,3

To generate the next row just subtract 1 from each number in the row vertically above. If the number is 1 then change it to n.
For FO fold order matrix you can generate each FO row from its corresponding CC row, as already shown above. If the first row for FO is known you can just circularly permute the FO numbers of the row positions above as (p-1) mod n = new p position for the that value.

Or more simply just move every number from the row above one position to the left, and put the leftmost number ffrom the row above into the rightmost position to complete the row.

Here is an example matrix for a 7_p with CC=1,5,2,7,3,6,4.

CC, FO n row matrix
7_p LP1=1,2,3,4,5,6,7(marks on the circles) 7_p LP2=1,3,5,7,2,6,4(new mark order CC1 to CC2)

7p CC1	twist	7p FO1	twist	7p CC2	twist	7p FO2	twist
1,5,2,7,3,6,4	+7	1,3,5,7,2,6,4	+7	1,3,5,7,2,6,4	+7	1,5,2,7,3,6,4	+7
7,4,1,6,2,5,3	-5	3.5.7.2.6.4.1	-5	7,2,4,6,1,5,3	-5	5,2,7,3,6,4,1	-5
6,3,7,5,1,4,2	-9	5,7,2,6,4,1,3	-9	6,1,3,5,7,4,2	-1	2.7.3.6.4.1.5	-1
5,2,6,4,7,3,1	-5	7,2,6,4,1,3,5	-5	5,7,2,4,6,3,1	-9	7,3,6,4,1,5,2	-9
4,1,5,3,6,2,7	+7	2,6,4,1,3,5,7	+7	4,6,1,3,5,2,7	+3	3,6,4,1,5,2,7	+3
3,7,4,2,5,1,6	-1	6,4,1,3,5,7,2	-1	3,5,7,2,4,1,6	-1	6,4,1,5,2,7,3	-1
2,6,3,1,4,7,5	+7	4,1,3,5,7,2,6	+7	2,4,6,1,3,7,5	+7	4,1,5,2,7,3,6	+7
M_{TT} total	+1		+1		+1		+1

Generating the 14 unique matrices for 7p
The matrix total twist, (M_{TT}) is always the same. This means that M_{TT} is always an invariant constant for any CC or FO matrix for any specific linear $\mathbf{A}_{CL=}${k1,L}. A total of 2n **different matrices** are possible for a linear $\mathbf{A}_{CL=}${k1,L}. For a prime that has no symmetry and is not its own mirror image each CC matrix generated from different FO>>CC starting rows will produce columns of row twists that are different, but the M_{TT} totals for each matrix are equal. For a link that is a single close link there is only one matrix because all 2n matrices are alike, just having rows in a different permuted order.

Each of the two fold order matrices, FO1 and FO2 has n rows each of which can be used as the first row for generating a new CC matrix of n rows. These 2n CC matrix can be called CC1,1, CC1,2, … CC1,n and CC2,1, CC2,2, … CC2,n. If a single nxn matrix of these columns of row twists for a set of n matrices is built, each row and each column of twists will add to the same total twist, M_{TT}. Below is a matrix of these n matrices with the n row twists built for the FO1 starting row of the 7_p matrix. Examination shows all rows to be unique with no circular permutations of other rows.

FO1 starting CC rows creating n 7x7 matrix
each with 7 row and column Matrix total twists with 7 twists per row.

		M_{TT}(matrix totals)
CC1,1 row twists	7,-5,-1,-9,3,-1,7	1
CC1,2 row twists	-5,7,7,-5,3,-5,-1	1
CC1,3 row twists	-9,-1,-5,7,11,-1,-1	1
CC1,4 row twists	-5,-1,-9,-1,-1,11,7	1
CC1,5 row twists	7,7,-5,-1,-5,3,-5	1
CC1,6 row twists	-1,-5,7,7,-1,3,-9	1
CC1,7 row twists	7,-1,7,3,-9,-9,3	1
Column totals	1, 1, 1,1, 1, 1,1	

FO2 starting CC rows creating n 7x7 matrix
each with 7 row and column Matrix total twists with 7 twists per row.

		M_{TT}(matrix totals)
CC2,1 row twists	7,-5,-9,-5,7,-1,7	1
CC2,2 row twists	-5,7,-1,-1,7,-5,-1	1
CC2,3 row twists	-1,7,-5,-9,-5,7,7	1
CC2,4 row twists	-9,-5,7,-1,-1,7,3	1
CC2,5 row twists	3,3,11,-1,-5,-1,-9	1
CC2,6 row twists	-1,-5,-1,11,3,3,-9	1
CC2,7 row twists	7,-1,-1,7,-5,-9,3	1
Column totals	1, 1, 1,1, 1, 1,1	

The nxn matrix represents a completed fold cycle about the x axis for each matrix. The column twists have the same total because they start with the next FO of the first FO matrix as the first CC of the next matrix. Each matrix represents a 180 degree rotation of the other matrix about the top left to bottom right diagonal and are equal in this sense.

Each of the row twists is an odd number or an even number for a given linkage(zero included as even). If the number of additions to calculate a row total twist is odd and n is odd then the matrix total twist for that linear $\mathbf{A}_{CL=}\{k_{1,L}\}$ cannot be zero. Thus 7 prime and 11 prime or any 7 + 4v prime where v=0,1,2... cannot have a zero M_{TT}.

Mirror Image and Axial Rotation

Seven prime has a matrix total twist of +1 so the mirror image, 7p, Mtt = -1. Since 7p has 14 unique matrices then 7p also has 14 matrices.
A mirror image CC is designated with an underline CC The mirror image of a CC code is the code

written in reverse thus **mirror image of <u>CC</u> = $CC_{((n+1)-p)}$ (ie. reverse positioning by writing the <u>CC</u> code in reverse order of the CC code, left to right becomes right to left).** Rotation about the x,y,z axis is designated CX, CY, CZ and leaves twist unchanged. CX does not change the code but reverses the binary coloring of the circles(ex. flat circles black on one side and white on the other). CY changes the code and reverses binary coloring. CZ changes the code to the same code as CY but retains binary coloring so we shall use it to derive distinct matrices. Then:
CZ = $((n+1) - CC)_{((n+1)-p)}$ ie. subtract each number of CC from (n+1) then reverse positioning to get the CZ code. Thus **<u>CZ</u> = (n+1) - CC** which is the same as:
<u>CZ</u> = $CZ_{((n+1)-p)}$.
The twist for all mirror code <u>CC</u> is always opposite to its image code CC, thus
t(CC) = -t(<u>CC</u>),
t(CC) = -t(<u>CZ</u>).
Any $\mathbf{A}_{CL=}\{k_{1,L}\}$ can be added to another $\mathbf{A}_{CL=}\{k_{1,L}\}$. We can add five prime to its mirror, 5p#<u>5</u>p, to produce a composite prime. Think of 5p as a single circle then link another 5p circle to it. The code of for this can be made by writing the 5p code then appending <u>5</u>p + 5 code to it on the right or left. Thus:

1,3,5,2,4,9,7,10,8,6 = 5p#<u>5</u>p with Mtt = 0, also,
1,3,5,2,4,10,9,8,7,6,16,17,18,19,20,14,12,15,13,11 = 10n#<u>10</u>n with Mtt = 0 because the last 10 integers are the mirror image of the first ten integers. An additional property of mirror image composition is if the added codes are interleaved then Mtt = 0. Thus 1,3,5,9,2,7,4,10,8,6 = 5p(#int1)<u>5</u>p with Mtt=0. Thus interleaving adds up to 2n-2 new linkages and unique matrices. Interleaving has been found to work only for exact mirror composition, not for FO#<u>CC</u>, or <u>FO</u>#CC composition combinations.

If we add a CC to its mirror image, <u>CC</u> we always get Mtt(CC + <u>CC</u>) = 0. By composing 5 prime with its mirror image in several different ways we get 16 unique matrices for n prime, 2 for each combination. We have:

CC#<u>CC</u>, **<u>CC</u>#CC,** we also have <u>CC</u>#FO, FO#<u>CC</u>, CZ#<u>FO</u>, <u>FO</u>#CZ, <u>CZ</u>#FO, FO#<u>CZ</u>,
CC#<u>CZ</u>, **<u>CZ</u>#CC,** CC#<u>FO</u>, <u>FO</u>#CC, CZ#<u>FZ</u>, <u>FZ</u>#CZ, <u>CZ</u>#FZ, FZ#<u>CZ</u>,
<u>CC</u>#CZ, **CZ#CC,** CC#<u>FZ</u>, <u>FZ</u>#CC, <u>CC</u>#FZ, FZ#<u>CC</u>,
<u>CZ</u>#CZ **CZ#<u>CZ</u>,**
where all CC and CZ, FO, FZ represent identical linkages(but in different code positions). All Mtt = 0. This may not exhaust all possibilities for unique matrices for the same linkage, for instance if some interleave compositions turn out to be identical linkages to the left/right ones. More codes can be added such as **CC#<u>CC</u>#CZ#<u>CZ</u> and CC#<u>CC</u>#<u>FO</u>#FO** for Mtt=0 by being careful to alternate added code lay overs. Always paying attention that total physical twist is produced as desired in the twist calculations for the coded matrices.
Composition of primes by thinking of each prime as a single circle allows more abstraction such as squaring a prime by composition. For instance 5p#^2 would be linked as (5p+(0*5))#(5p+(2*5))#(5p+(4*5))#(5p+(1*5))#(5p+(3*5)) (multipliers work out as code adders. 0,10,20,5,15, ie 5p *circles are linked as* **1,3,5,2,4** which is the code for 5p) meaning that each prime is laid over so that no two of the five 5p *circles* can be close. This is like a code of codes.

Adding rows to produce zero twist
If Mtt is zero and two rows have equal but opposite twists, one positive and the other negative, then

adding the two rows to produce a composite linkage always produces zero Mtt for that combined linkage.

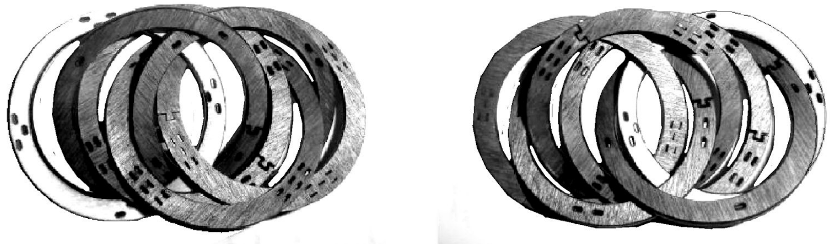
Linkage Total Twist, L_{TT} and Link Total Twist, L_{KT}
For linear, or $\mathbf{A}_{CL=}${k1,L} The matrix total twist, M_{TT}, is the sum of all possible twists for a linear code cycle. Each matrix has n rows so dividing by n gives the linkage average twist $L_{TT}=M_{TT}/n$.

Each circle can then be given an average twist called the link average twist(twist per individual circle), $L_{KT} = M_{TT}/n^2$ or L_{TT}/n.
When we calculate the M_{TT} twist for a max twist linkage, where every link twists in the same direction (a close link of n links) we find that as links are added the twist added as n increases tends to rise slowly toward a straight line. Twist data can be gotten using the matrix calculation methods detailed in ref. [6]. The L_{KT} for max twist(close) linkages n=10 to n=20 links is:

10	11	12	13	14	15	16	17	18	19	20
1.2	1.36	1.52	1.69	1.86	2.02	2.19	2.35	2.5185	2.6842	2.85

For $\mathbf{A}_{CL=}${k1,L} The increase in twist per circle going from n=15 to n=16 is approximately 0.1653 and this increment between n=19 and n=20 is 0.1658 or an additional increment of 0.0001 when adding 1 link then link average twist increases as L_{CT} =0.16n approximately for this example, tending to a straight line increment. If this link twist continues to increase at a small but steady pace it must eventually become large for each circle in the linkage as n gets large. Each added link would add considerably to the L_{TT}, linkage total twist. Each link of a 10000 link A_{CL}k1 would add about L_{KT}=0.16*10000=1600 twists per per link when adding one link. Therefore total linkage twist, L_{TT}=1600*10000= 1.6x10^7 approximately.

Binary Coloring
Figure 3 shows two equal linear 6 primes with a binary coloring. The circles are colored perpendicular to the large radius R, light gray on one side and dark grey on the other side. Twist is based on how the circles lay over each other in a linear array. A circle can only be dark side or light side up. If the circles are marked then a given marked binary coloring should always produce the same twist. This is true no matter how the circles are re-positioned linearly. In the two example photos we have the circle marked ‘three’ colored light grey and the others are colored dark gray, so all such marked coloring’s must produce the same twist calculation. Of course if you flip the entire array over so there is one dark gray and 5 light grey, the twist calculation is still the same since no folding has taken place, leaving the colored marking constant. Each circle, when laid over the opposite way adds or subtracts the same amount of twist no matter where it is located in the linear array.



a b

Figure 3

Demonstration of t(CC),=t(FO), Twist Invariance for $A_{CL=}\{k1,L\}$

It is necessary to rearrange the CC and FO so that they exchange places with each other without folding any of the circles, CC becomes FO and FO becomes CC. Refer to Figure 4 for a visual illustration of one way to do this. Grasp the uppermost(z axis) links with your fingers on left and right of the y axis line of links and pull them apart along the x axis and then move them together along the y axis thereby transferring the linear array of circles to line up along the x axis. Because of the way CC is generated by writing the z level along x and FO is generated by writing the x level along z then CC and FO can change places with each other. Since this operation does not involve any folds of the circles the twist of CC must exactly equal the twist of FO. This should always be possible since the circles are first free to move along one axis across their D diameters, and then along the other axis across their D diameters. Done in practice with several physical links, it always works, if done carefully and all links are involved in the pull apart along one axis then move together along the other axis.

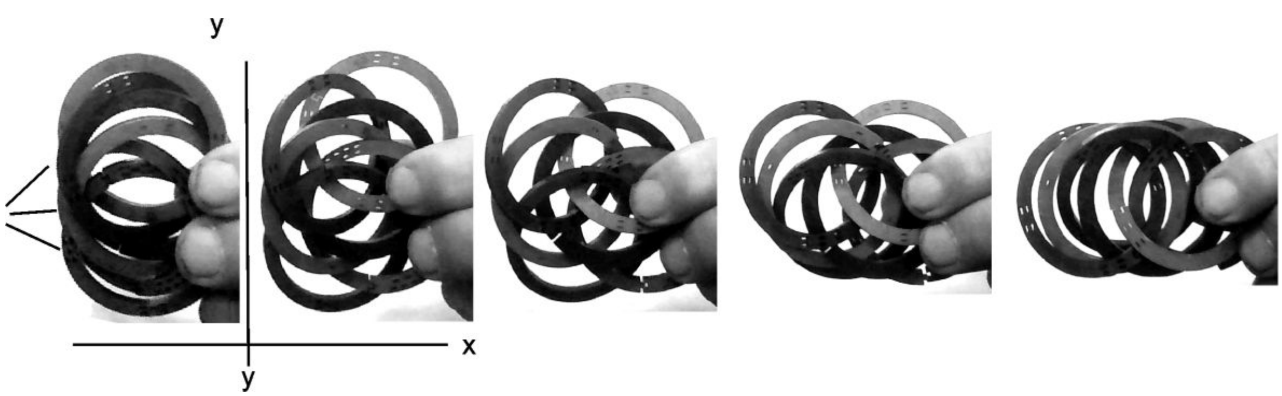


Figure 4

Matrix Total Twist Invariance , M_{TT}, for $A_{CL=}\{k1,L\}$

The set of folds that generate each row of a CC matrix always complete one cycle after n folds generating n rows of the nxn CC fold matrix. This holds true for all linear linkage codes $A_{CL=}\{k1,L\}$. Each pair of circles have two layover positions, shown in Figure 1a and 1b. No matter how the circles are rearranged in a line a complete set of n folds will produce a complete set of pair comparisons since folding can only occur as a simple permutation process. Thus if there are more circle pairs with code

CCi>CCj with j>i (positive twist) than with code CCi>CCj with j<i (negative twist) then more pair comparisons with positive twist will occur than pair comparisons with negative twist. The entire set of pair comparisons for nxn matrix must always be the same making the matrix total twist invariant for all linear arrangements of any $\mathbf{A}_{\mathbf{CL=}}${k1,L}.

Toroidal Rotation for Linear A_{CL},k1 Links

It has been found that if a $\mathbf{A}_{\mathbf{CL=}}${k1,L} linkage is made of thin circles and held flat in a circular and symmetrical arrangement it can appear to turn inside out about its toroid axis. The circles all move simultaneously to opposites sides of the surrounding torus space. This has been found to work for some primes and works well if the n circles are a single close link. It is conjectured to be similar to the process of changing CC to FO as in Figure 4 where CC to FO woul.d be isomorphic to a ½ toroidal rotation. Of course with more and more circles we need them to be thinner and thinner. The twist remains the same as the linkage rotates, meaning that it is always in the same flat arrangement even though the circles are moving about with respect to each other.

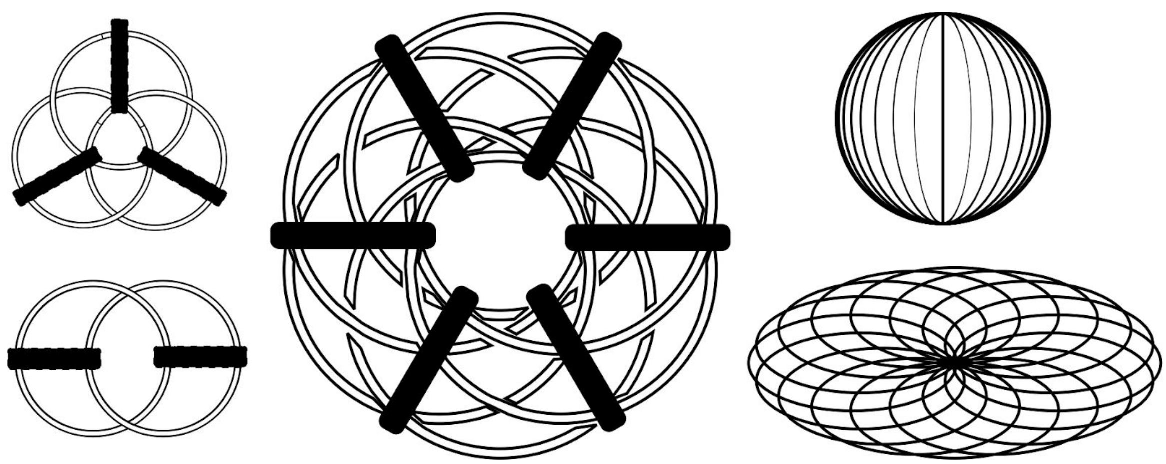


Figure 5

Symmetrical Node Puzzles for Linear A_{CL}={k1,L}

It is possible to arrange an $\mathbf{A}_{\mathbf{CL=}}${k1,L} linear link in a symmetrical flat circle. By passing symmetrical groups of node crossings thru holes made in some bars, a puzzle like device can be made. The simplest, which I call diaka,(in sequence 3=triaka, 4=tetraka, 5=pentaka, etc.) consists of only two

circles and does not use any node crossings. For more than two circles, if the holes in the bars are big enough to allow for some clearance and are spaced just right then you can rotate the bars continuously in a group about a circular torus axis. When doing this rotation the circles appear to turn inside out too, but instead they just move from side to side through the torus space thereby making a nice optical illusion puzzle. The motion seems hard to imagine, especially for puzzles with a greater number of circles n. Of course the 6 circle structure shown, at the left in Figure 5 (called 'Hexaka') cannot be perfectly flat when in the flat position because the circles have thickness and pass over each other at the nodes. When the bars rotate 90 degrees about the toroid axis the figure emulates a spherical shape with holes at the poles. The idea can be extended. Imagine that a large number of very thin $\mathbf{A}_{\mathbf{CL=}}${**k1,L**} circles are placed closely in a flat circular array as seen at lower right in Figure 5. If some method of holding the nodes symmetrically existed this circle shape could be folded into a spherical shape as seen at top right in Figure 5. Continuing the folding would return it to a flat circle shape. The flat circle shape has a point in the center where a concentration of all the circles pass over each other while the spherical shape has two opposed polar nodes where this happens.

Non Linear A_{CL}={k1,r} or Rogue Links
A non linear or Rogue $\mathbf{A_{CL=}}${k**1**,**r**} cannot be manipulated so that the centers of its circles all lie on a straight line. Figure 6, presented both as a model and as a plane projection, consists of 7 circles and is an $\mathbf{A_{CL=}}${k**1**,**r**}, Rogue link. This appears to be the smallest possible Rogue link. A rogue link also satisfies the definition of a non linear prime where no two links can be made close. This rogue link is designated 7r. It always has the centers of its circles in a non linear arrangement.

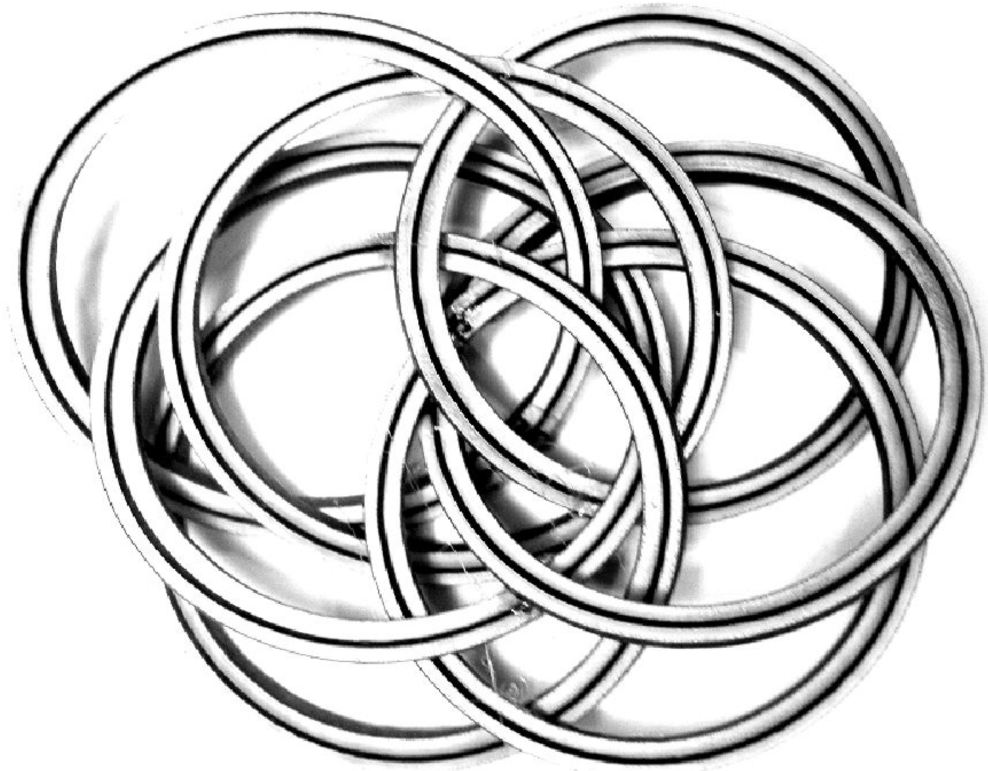


Figure 6

Rogue links must exist. From the above discussion about linear linkages we know that any linear prime can have up to 2(n^2) unique permutation arrangements. It is possible to arrange n integers in n! ways. This means that n!-2(n^2) ways exist of trying to rearrange the circles of a prime so that none of the circle centers lie in a straight line, otherwise more than 2(n^2) linear arrangements would be possible. Therefore n!-2(n^2) of the possible attempted linear re-arrangements are candidates for making an n+1 rogue link, with this number increasing exponentially as n increases since n!-2(n^2) increases much faster than 2(n^2). You can create a Rogue prime by putting a linear prime into one of these non linear attempted construction code arrangements. Then add a new link by linking it through all the circles very randomly. If this new linkage can be manipulated to form a linear link you can remove the link then try adding it in a different way. Repeat this process until you get a rogue prime. Once you have created a rogue prime this way way you can always make a larger one by adding another link to it in the same way. This method was attempted with a 5 prime but always resulted in a 6 prime not a Rogue prime. Then when it was attempted with a 6 prime a Rogue link, 7_r, was the result shown in Figure 6. A possible construction code for Rogue primes is to arrange the circles in a left to right manner.

Identify each circle in left to right fashion then list the twist of all ordered pairs of circles.

An open question is to find an example of the smallest number of circles required to make a lock link . None of the circles of a lock link can be folded or laid over. A true lock link should allow d/D to become smaller and smaller and stay locked.

Self organizing of close links

Close links always gather together in separate twisted groups. As the circles are made thicker this effect becomes more pronounced. The reason has to do with geometry. Close circles in a group occupy the least volume when they are close to each other in code order 1,2,3,4,…,n. They tend to exclude links that twist the opposite way. This is a natural kind of self organization in $\mathbf{A}_{\mathbf{CL}}=\{k\mathbf{1},\mathbf{L}\}$ architectures.

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