Maximally linked Circles With Invariants

Presenting some new integer invariants

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Have you ever wondered in how many different ways, *w*, you can link a number of identical circles, n, together so that every circle links thru every other circle? To the writers surprise as n increases *w* becomes more and more difficult to determine. Since the circles are identical it is helpful, in some cases, to put an identifying mark on each circle, such as '1', '2', '3'...'n' . If you can arrange the marked circles in a **linear** marked order and you are able to rearrange the marked circles so that they are in a different linear order then you know that at least two linear marked orderings are possible. Amazingly when it is possible to arrange the circles in a linear order they have very well behaved mathematical properties no matter how different two linkages of n circles are. In what follows you will discover this enables rapid calculation of some basic properties of any size linear linkage, just using a simple integer construction code. Beyond this it is an incredible fact that when you get to a linkage of seven or more circles it is possible to make the linkage in such a way that a linear arrangement is not possible. This is called a **Rogue link**. To add insult to injury it turns out that there are exponentially more Rogue links than there are nice linear links. All these results are quite simple to explain, just read on.

Define '**All circle links**' $\mathbf{A_{CL}}$ as the set of identical linked circles. The writer is not aware of a mathematical symbol for All circle links so for purposes of this paper we will use the notation $\mathbf{A_{CL}}$. For research purposes our $\mathbf{A_{CL}}$'s are made by linking identical, solid, circular tori together so that every circle links thru every other circle. Thus $\mathbf{A_{CL}}$ without delimiters denotes the set of all possible linked circles. For our purposes it does not include any unlinked circles except for the $\mathbf{A_{CL}}$ set of a one circle. A circular torus has major and minor diameters, **D** and **d**. There is no proof given here, but it is assumed that the results for linked circles with d>0 will hold as the circles are made thinner and thinner (D/d made larger) so it is OK to call them circles. Only a certain maximum, n, of these circles can be linked so that every circle links thru every other circle. This maximum, determined by the ratio D/d, has an approximate value of n=(2D/d)-2, so n can be made larger by making d smaller or D larger. The reason for this maximum is because as n increases there is less and less room for n-1 circles to pass thru a single circle due to their thickness or small diameter d.

Given n links, each linking to n-k=q links , then if every link links once through every other link then then it does not link through itself, so k=1, and q=n-1. Since q varies with n but k is constant $\mathbf{A_{CL}}$,k_1 shall be the set of all maximally linked circles. For instance k2 would mean every circle does not link through exactly 2 circles. Here the $\mathbf{A_{CL}}$,k_1 set notation for k1 means k=1. An example member of $\mathbf{A_{CL}}$,q2 would be a circular chain of 4 or more circles where each circle links to its two neighbors. If q is constant k can vary but we will only consider $\mathbf{A_{CL}}$,k_1 systems here. We will use the same shorthand for several more $\mathbf{A_{CL}}$ variables below. This way we can talk about different kinds of $\mathbf{A_{CL}}$ sets because only the subject variables need to appear. The Hopf link consists of two linked circles and is the simplest nontrivial link[5]. We only encounter the Hopf link for n=2 and do not need it beyond that. For linear $\mathbf{A_{CL}}$,k_1 and if the circles are thin enough you can lay them out on a surface in a projected **flat linear array**, defined as having the center of each circle lie on a straight line. When k is 1 and the system is linear the circles can often be rotated in a toroidal motion about the circular toroid axis. Much of this discussion is about the properties of linear examples from the set of $\mathbf{A_{CL}}$,k_1 links.

An $\mathbf{A_{CL}}$,k_1 '**close link**' is defined in this paper as a linkage of two or more circles that are linked so that each circle twists in the same direction and so that each pair of circles in the close link can be manipulated and made to touch around their circular toroidal surface. We need this definition because as will be seen below links of five or more circles can be made where no close links are possible, called a **prime link**.

Close links can be seen to twist around each other as shown in **Figure 1**. The smallest close link is two linked circles. By laying two circles closely over one another in two ways you can see the circles as either having one positive twist or one negative twist [1]. If more circles are added to a close link the twist becomes more 'locked in' but can still vary by laying the circles over in different ways. Close links of n links allow the n circles to be placed in any marked order or n factorial possible ways. The absolute total twist is maximum when the entire $\mathbf{A_{CL}}$,k_1 link of n circles is a single close link. A simple way of calculating the twist for any linear flat projection $\mathbf{A_{CL}}$,k_1 link will follow below. For an $\mathbf{A_{CL}}$,k_1 linkage of n circles **u** is the total number of circles that cannot be made close to any other circle in the $\mathbf{A_{CL}}$,k_1.

$\mathbf{A_{CL}}$,k1,u0,m1,n2=1 **$\mathbf{A_{CL}}$,k1,u0,m1,n14=2**

Figure 1

Close links of different twists can be linked together in various ways. This variable is called **m** and is the number of different close links in an $\mathbf{A_{CL}}$,k_1, so we have the set **$\mathbf{A_{CL}}$,k1,u,m**. For instance you could have a link where 5 close links twist one way linked to another where 3 close links twist the opposite way making m=2, u=0. It may seem that u and m are redundant used together, but for many links they are not redundant. The next variable is the number of circles in the linkage called **n** and we have the set **$\mathbf{A_{CL}}$,k1,u,m,n**. For our purposes we can let this set equal the number of elements it contains, *w*, as seen in Figure 1 where the left pair shows the only way to link two circles. The close link at right in Figure 1 can be made with either positive or negative twist so it has two elements. It is evident that if m=0 then u=n. This is called an **$\mathbf{A_{CL}}$,**k1 prime, mentioned above, where no pair of links are close. Finally we have the total twist of the flat projection, 0, +t or -t. All of these variables, even if you include twist can specify one or more different links, w>=1. This notation provides a handy way of proposing different properties to investigate. Twist for any flat projection can be different depending on how the circles lay over each other (as in Fig. 1), but it has well behaved invariant properties for linear links which will be revealed below.

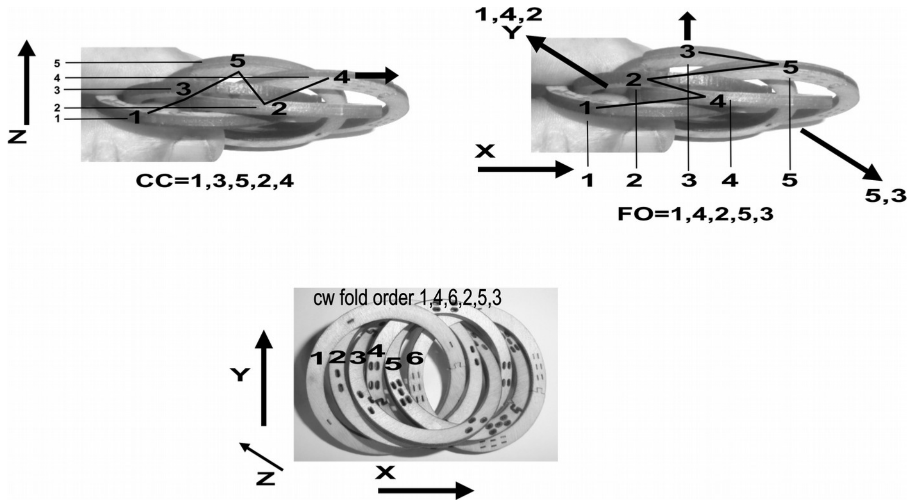


Figure 2

Figure 2 top left and right shows a **linear 5 prime linkage** of five circles. Five prime is the smallest possible prime so that u=n=5. Refer to the top left part of Figure 2 to see how a **construction code**, **CC**, is written for this linkage. The leftmost circle is in the level 1 z axis position. The next to leftmost circle is in the level 3 z axis position and continuing we can write the level, or z axis position of each circle moving from left to right along the x axis as 1,3,5,2,4. This will be the first row of a **CC matrix**. It is a convention to put the left circle in the level 1 position when writing the first row of a CC matrix. These matrices will be discussed below.

The top right part of Figure 2 shows how the **fold order**, or **FO** is written based on going along the x axis and writing the x position of the first circle in z level 1, then write the next x position of the circle in z level 2 and so on. This is the order of folding of the circles when folding each circle one at a time down or toward you. The bottom part of Figure 2 shows the FO for a 6 prime with marked circles. As you can see you would first be able to fold down(about the x axis) the circle in position one in the listed fold order then the next circle able to fold is in the listed position 4 and so on. A prime requires that u=n so that no pair of circles can be made close to each other. This means that any two adjacent integers in the construction code or CC must have a difference greater than 1. This includes the first and last integers since these are adjacent by circular permutation.

The twist of a linear **A**CL,k1 can easily be calculated from the CC. The twist also has exact invariant properties. Twist can vary depending on how the circles lay over each other, so twist invariance depends on lay over position or a sum of lay over positions.

Calculate the twist of a construction code, CC, by starting at the second number from the left and working to the right. If the second number is greater than the first it adds +1 if less it adds -1. Now compare the third number to the second and first number and add or subtract a one for each if greater or less. The fourth number is compared to the third, second and first adding a 1 or -1 each time. Continue this for the n circles. This produces a twist number for each circle from circle 2 to circle n by summing the 1's found going to the right in each case. Summing these twist numbers gives the total twist of that positional array, or CC of circles. Here is the method in action using a 5 circle system:

Given prime CC=1,3,5,2,4 (and n=5 since there are 5 circles)

3>1, t=1 second circle twist is 1
5>3, t=1
5>1 t=1 third circle twist is 2
2<5, t= -1
2<3, t= -1
2>1, t=1 fourth circle twist is -1
4>2, t=1
4<5, t= -1
4>3, t=1
4>1, t=1 fifth circle twist is 2

This makes the total twist 1+2+-1+2 =+4 for this group of circles in this position. This calculation can be performed in several different ways , comparing left to right, right to left, and so forth, and each way will always produce the same total twist. This is because all possible left to right pair comparisons are performed.

It has been found that this twist calculation for any linear $\mathbf{A}_{CL}$,k1 works for any set of positive sequential numbers 1 thru n.   If you flip the array over left to right and then determine its new construction code, CC, and recalculate the twist it will be the same. For instance rotating CC=1,4,6,2,5,3 180 degrees about the z axis renumbers as CC= 4,2,5,1,3,6 and each of these gives a twist of +3. If the numbers of a CC row are listed in reverse order a mirror image linkage results and just the sign of the calculated twist reverses. Since five prime is its own mirror image reversing the CC or FO numbering results in the same linkage and its total matrix twist(matrix twist explained below) is zero. The twist calculation presented here is based on the accepted twist for two linked loops in a planar projection as can be found in reference number [1]. You will also get the same twist when you derive the fold order, FO from the CC. then calculate the twist for the FO. The FO is complementary to the CC in always producing the same total twist and in other ways. Fold order can be derived from the CC listing as follows.

Given the CC=1,3,5,2,4 then fold order, FO can now be listed as 1,4,2,5,3 as follows:
1 since 1 in the CC is in position 1
4 since 2 in the CC is in position 4
2 since 3 in the CC is in position 2
5 since 4 in the CC is in position 5
3 since 5 in the CC is in position 3

Note that for certain number sequences CC sometimes exactly equals FO. This is the case for CC=1,4,6,2,5,3=FO=1,4,6,2,5,3

Every CC, FO position of a linear link can manipulated so that the old CC becomes the new FO and the old FO becomes the new CC. Therefore FO can be treated as if it were a CC and then calculate its twist.

Given FO=1,4,2,5,3 (from CC=1,3,5,2,4)
4>1, t=1 second circle twist is 1

2<4, t= -1
2>1, t=1 third circle twist is 0
5>2, t=1
5>4, t=1
5>1, t=1 fourth circle twist is 3
3<5, t= -1
3>2, t=1
3<4, t= -1
3>1, t=1 fifth circle twist is 0

Total twist is therefore =+4 and this is exactly the same as the CC that generated this FO. Thus t(CC)=t(FO) and is an invariant. Since we can treat FO as if it were a CC we always have two different linear CC, where FO(CC1)=FO1 and FO1=CC2, and FO(CC2)=FO2 and FO2=CC1 exactly. This is only works if the leftmost CC code number =1, generally the first row in the **fold matrix**. Thus the FO of a CC represents another linear arrangement of the CC that can be gotten by physically moving the circles. In this sense FO and CC are complementary to each other. In the case of a close link CC1=CC2 so only one CC=FO exists. For **A_{CL}**k1 primes, CC1 may or may not equal CC2. You can make a prime where CC1=FO1=CC2=FO2. This makes the two CC matrices identical. In that case one way to tell that two linear arrangements are possible is by numbering or marking the *individual* circles, then rearrange the circles to show that two different marked arrangements exist that are not just a circular permutations of each other. A close link of x marked links can have its circles put in any linear marked order, giving x! possible marked close arrangements for a close link of x links so all the circles of a close link can be given the same mark.

As mentioned above u is the number of circles that are not close to any other circle in the, **A_{CL}**k1. This motivates a search for when u=n meaning an **A_{CL}**,k1 which has no close circles, or a prime link. The smallest link for which this is possible is when n=5 designated as 5_p, five prime. Since five prime is its own mirror image and no other 5_p exists it is denoted **A_{CL}**,k1,u5,m0,n5=1. You can have a 5_p that has close links added to it so that you could have **A_{CL}**,k1,u4,m2,n12,5_p=x. This set having a 5_p means that each member of the set contains a 5_p as follows: If you reduce all the close links to one link you always stop at 5_p.

Rules for a linear A_{CL} k1 to be prime: 1. The absolute difference between adjacent integers of the CC code must be greater than 1, including the leftmost and rightmost integers as these are also adjacent by circular permutation. 2. The **A_{CL}**,k1 cannot be a composite of two or more individual prime links linked together. 3. An ordinary linear **A_{CL}**,k1 n prime can be arranged linearly with only two different circular linear permutations of the marked circles. 4. A flat linear arrangement must be possible. If any close links existed Rule 3 would be broken. These rules are merely a recipe and have are somewhat redundant.
In rule 2 for instance p5#p5 (link a 5 prime to another 5 prime) to give CC=1,3,5,2,4,6,8,10,7,9. It has no close links but the two prime links can be independently rotated about y and z axes and folded over allowing extra linear permutations breaking Rule 3. It is a composite prime link and/or could also be called close prime link.

A method for building an n+1 prime

Two circles at a time are available to fold over in a linear **A_{CL}**,k1 prime or non prime arrangement.

These are the circles with CC level 1 and CC level n and are called naked links. The two naked links represent the two directions of folding about the horizontal x axis. Folding in one direction n times for an n circle $\mathbf{A}_{CL}$,k1 completes one fold cycle in that direction. So folding in either direction gives exactly the same set of n fold arrangements but in a different order. Of these n-4 produce ends with no naked end links. When adding a circle to make an n+1 prime there can be no naked end link as the naked end link would form a close link with the new added link. Since the primes generally have exactly two unique CC's this gives 2(n-4) possible positions having no naked end links. Thus 5_p has two no naked end link fold positions that can be used to build 6_p. Due to symmetry both of these are equivalent providing only one way to make 6_p. Then 6_p has four no naked end link fold positions and so on. The other way to find primes is to analyze the number of different ways you can arrange n sequential numbers according to the prime rule 1. It is not known if this is sufficient. For instance could moving circles around in a physical linkage bring two of them together into a close link after building the link according to a proposed CC code?

Linear Permutation Matrices `

The marked circles show how the linear primes limit themselves to only two kinds of circular permutations and are helpful to record and reconstruct specific arrangements and permutations.

An nxn matrix is our goal. To generate the next matrix row of a CC of n links change 1 to n from the row above and subtract 1 from the other numbers of the row above for the next row CC. Continue this process for each row. This is the case since the next row CC is gotten by folding the level 1 circle down to become the level n circle.
For FO fold order matrix you can generate each FO row from its corresponding CC row, as already shown above. This can be defined as an operation of FO on CC to get each row, FO(CC1,1) = FO1,1 FO(CC1,2) = FO1,2 …
Here is an example matrix for a 7_p with CC=1,5,2,7,3,6,4.

CC, FO n row matrix

7_p LP1=1,2,3,4,5,6,7(circle markings)				7_p LP2=1,3,5,7,2,6,4(circle markings)			
7p CC1	twist	7p FO1	twist	7p CC2	twist	7p FO2	twist
1,5,2,7,3,6,4	+7	1,3,5,7,2,6,4	+7	1,3,5,7,2,6,4	+7	1,5,2,7,3,6,4	+7
7,4,1,6,2,5,3	-5	3.5.7.2.6.4.1	-5	7,2,4,6,1,5,3	-5	5,2,7,3,6,4,1	-5
6,3,7,5,1,4,2	-9	5,7,2,6,4,1,3	-9	6,1,3,5,7,4,2	-1	2.7.3.6.4.1.5	-1
5,2,6,4,7,3,1	-5	7,2,6,4,1,3,5	-5	5,7,2,4,6,3,1	-9	7,3,6,4,1,5,2	-9
4,1,5,3,6,2,7	+7	2,6,4,1,3,5,7	+7	4,6,1,3,5,2,7	+3	3,6,4,1,5,2,7	+3
3,7,4,2,5,1,6	-1	6,4,1,3,5,7,2	-1	3,5,7,2,4,1,6	-1	6,4,1,5,2,7,3	-1
2,6,3,1,4,7,5	+7	4,1,3,5,7,2,6	+7	2,4,6,1,3,7,5	+7	4,1,5,2,7,3,6	+7
M_{TT} total	+1		+1		+1		+1

Matrix Total Twist Invariance

The matrix total twist, (M_{TT}) is always the same. This means that M_{TT} is always an invariant constant for any CC or FO matrix for any specific linear $\mathbf{A}_{CL}$,k1. A total of 2n **different matrices** are possible for a linear $\mathbf{A}_{CL}$,k1. For a prime that has no symmetry and is not its own mirror image each CC matrix generated from different FO starting rows will produce columns of row twists that are different, but the

M_{TT} totals of each column, for each matrix will be equal. For a link that is a single close link there is only one matrix because all 2n matrices are alike, just having rows in a different permuted order.

Each of the two fold order matrices, FO1 and FO2 has n rows each of which can be used as the first row for generating a new CC matrix of n rows. These 2n CC matrix can be called CC1,1, CC1,2, … CC1,n and CC2,1, CC2,2, … CC2,n. If a single nxn matrix of these columns of row twists for a set of n matrices is built, each row and each column of twists will add to the same total twist, M_{TT}. Below is a matrix of these n matrices with the n row twists built for the FO1 starting row of the 7_p matrix. Examination shows all rows to be unique with no circular permutations of other rows.

FO1 starting CC rows creating n 7x7 matrix
each with 7 row total twists.

		MTT(matrix totals)
CC1,1 row twists	7,-5,-1,-9,3,-1,7	1
CC1,2 row twists	-5,7,7,-5,3,-5,-1	1
CC1,3 row twists	-9,-1,-5,7,11,-1,-1	1
CC1,4 row twists	-5,-1,-9,-1,-1,11,7	1
CC1,5 row twists	7,7,-5,-1,-5,3,-5	1
CC1,6 row twists	-1,-5,7,7,-1,3,-9	1
CC1,7 row twists	7,-1,7,3,-9,-9,3	1
Column totals	1, 1, 1,1, 1, 1,1	

FO2 starting CC rows creating n 7x7 matrix
each with 7 row total twists.

		MTT(matrix totals)
CC2,1 row twists	7,-5,-9,-5,7,-1,7	1
CC2,2 row twists	-5,7,-1,-1,7,-5,-1	1
CC2,3 row twists	-1,7,-5,-9,-5,7,7	1
CC2,4 row twists	-9,-5,7,-1,-1,7,3	1
CC2,5 row twists	3,3,11,-1,-5,-1,-9	1
CC2,6 row twists	-1,-5,-1,11,3,3,-9	1
CC2,7 row twists	7,-1,-1,7,-5,-9,3	1
Column totals	1, 1, 1,1, 1, 1,1	

This is an interesting twist invariant. The reason the matrix total twist is invariant is because a single matrix represents a completed fold cycle about the x axis for each possible matrix. The column twists have the same total because they start with the next FO of the first FO matrix as the first CC of the next matrix. Each matrix represents a 180 degree rotation of the other matrix about the top left to bottom right diagonal and are equal in this sense. Two different linear $\mathbf{A}_{CL}$,k1 can have the same M_{TT} but may have few other similarities unless there is some specified relationship. For instance some n prime M_{TT} can equal zero and so can the M_{TT} of an n circle link having close links with opposing twists.

Note that each new CC row is generated by a 180 degree fold of the topmost link about the x axis while each new FO can be generated by folding the leftmost link of each CC row 180 degrees about the y axis to become the rightmost link. Of course this could be reversed and the bottom and rightmost links could be the folding convention but the mathematical end results would be exactly the same. Another interesting fact about the row total twists is that each of the row twists will either be an odd number or an even number for a given link. This also means that if the number of additions to calculate a row total twist is odd and n is odd then the matrix total twist for that linear ACLk1 cannot be zero. It can be as small as 1 or -1 but not zero. Thus 7 prime and 11 prime or any 7 + 4a prime where a=0,1,2... cannot have a zero M_{TT}.

Linkage Average Total Twist, L_{TT} and Average Total Twist per circle, L_{CT}

For linear A_{CL}k1 The linkage total twist is the matrix total twist, M_{TT}, divided by n or the total number of circles in the linkage. Each matrix has n rows so dividing by n gives the average twist, defined as the link twist, per row. We have $L_{TT}=M_{TT}/n$

Each circle can then be given an average twist called the circle total average twist, $L_{CT}=M_{TT}/n^2$ or L_{TT}/n. When we calculate the M_{TT} twist for a max twist linkage, where every circle twists in the same direction (a close link of n circles or m=1) we find that as circles are added the twist added as n increases tends to rise slowly toward a straight line. A graph for this twist curve can be approximated using the matrix calculation methods detailed here. The L_{CT} for max twist up to 20 circles is:
2 is 0, 3 is 0.1111.., 4 is 0.25, 5 is 0.4, 6 is 0.555.., 7 is 0.71, 8 is 0.875, 9 is 1.03703, 10 is 1.2, 11 is 1.36, 12 is 1.52, 13 is 1.69230..., 14 is 1.857, 15 is 2.0222..., 16 is 2.1875, 17 is 2.35294... 18 is 2.5185..., 19 is 2.6842..., 20 is 2.85

For linear A_{CL}k1 The increase in twist per circle going from n=15 to n=16 is approximately 0.1653 and this increment between n=19 and n=20 is 0.1658 or an additional increment of 0.0001 when adding 1 circle thus tending to a straight line increment. Rounding down a very conservative estimate is L_{CT}=0.16n. Thus if this twist continues to increase at this small but steady pace it must eventually become large for each circle in the link as n gets large. Each added circle would add considerably to the L_{TT}, link total twist. Each circle of a 10000 link A_{CL}k1 would produce, L_{CT}=0.16*10000=1600 twists per circle. Therefore total link twist, L_{TT}=1600*10000=1.6x 10^8 approximately, and M_{TT}=1.6 x 10^12 approximately.

Binary Coloring Lay-Over Twist Invariance

Figure 3 shows two equal linear 6 primes with the circles colored light gray on one side and dark grey on the other, called a binary coloring. Since twist is strictly based on how the circles lay over each other in a linear array, then a specific marked binary coloring should always produce the same calculated twist result. If the circles are kept dark grey side up this link will always have a twist of +3 no matter how the circles are permuted linearly. In the two example photos we have the circle marked ‘three’ colored light grey and the others are colored dark grey, so all such marked colorings must produce the same twist calculation. Of course if you flip the entire array over so there is one dark grey and 5 light grey, the twist calculation is still the same since no folding has taken place, leaving the colored marking constant. This means that each circle, when laid over the opposite way adds or subtracts the same amount of twist no matter where it is located in the linear array. These observations can be used to help prove the twist invariance properties.

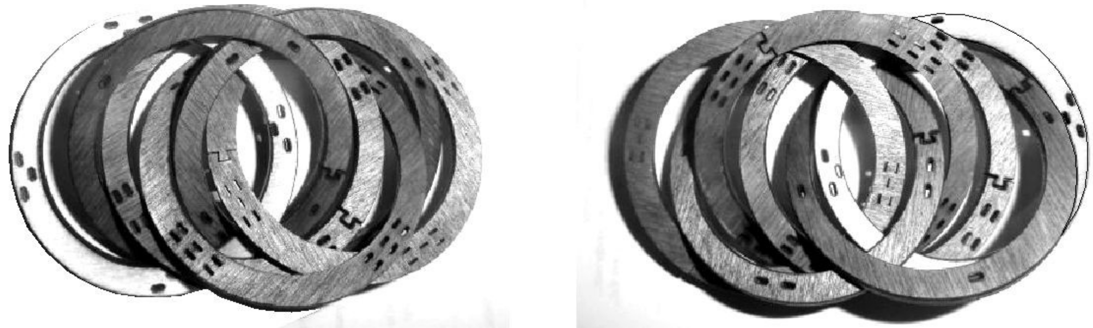


Figure 3

Proof of t(CC),=t(FO), Twist Invariance for Linear $A_{CL,}$k1

It is necessary to rearrange the CC and FO so that they exchange places with each other without folding any of the circles. The CC1 becomes FO2 and FO1 becomes CC2. The two codes, CC and FO just exchange places One way to obtain such a rearrangement for a linkage is to grasp the upper links with your fingers on both sides of the long portion of the link and pull them apart along one axis and then press them together along the other axis thereby transferring the linear array of circles to line up along the other axis. This process is shown by the group of five illustrations in Figure 4. Because of the way CC is generated by writing the z level along x and FO is generated by writing the x level along z then CC and FO exchange with each other during this operation. Since this operation does not involve any folds of the circles the twist of CC must exactly equal the twist of FO. This should always be possible since the circles are first free to move along one axis across their D diameters, and then along the other axis across their D diameters. Done in practice with several physical links, it always works, if done carefully and all links are involved in the pull apart-transfer. However for more complex links, such as composite primes, different groups of links in the different composing primes must be taken into account and pulled in their own groups. At any rate the writer presents this as an initial visual proof.

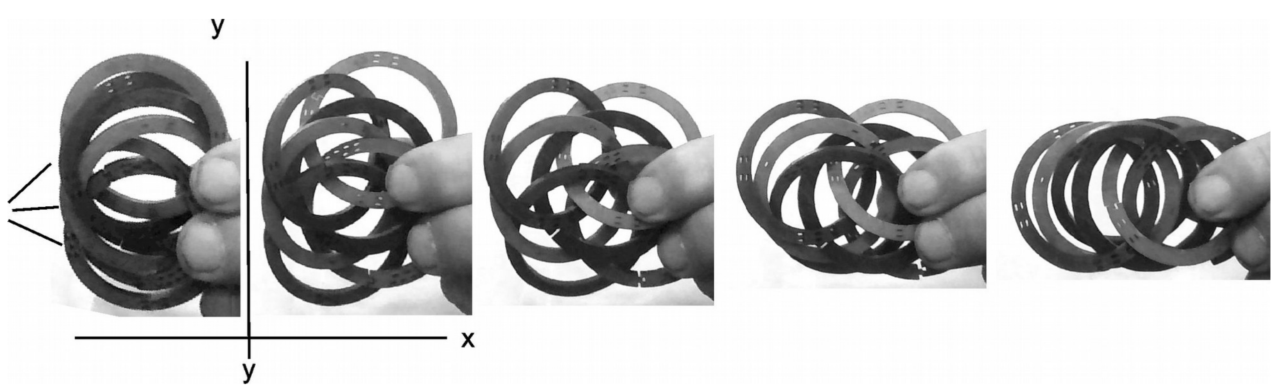


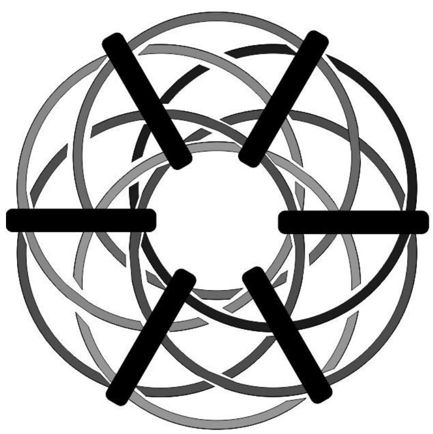
Figure 4

Proof of Matrix Total Twist Invariance , M_{TT}, for Linear A_{CL},k1

The set of folds that generate each row of a CC matrix take the binary coloring through a complete set of n marked colored projections for each CC matrix. As shown above, position of the circles does not affect twist if the marked coloring is the same for two different flat projections. This being the case the M_{TT} has to be constant for each of the 2n CC matrices.

Toroidal Rotation for Linear A_{CL},k1 Links

It has been found that if a linear A_{CL}k1 linkage is made of thin circles and held flat in a circular and symmetrical arrangement it can appear to turn inside out about its toroid axis. The circles all move simultaneously to opposites sides of the surrounding torus space. This has been found to work for some primes and is thought to work for many different kinds of links. It is conjectured to be similar to the process of changing CC to FO as in Figure 4 where CC to FO would be isomorphic to a ½ toroidal rotation. Of course with more and more circles we need them to be thinner and thinner. The twist remains the same as the linkage rotates, meaning that it is always in the same flat arrangement even though the circles are moving about with respect to each other. It is an open question to prove or disprove if this always works for linear linkages. In some cases various circles would probably need to reside in and rotate about nested tori for some kinds of linear links.



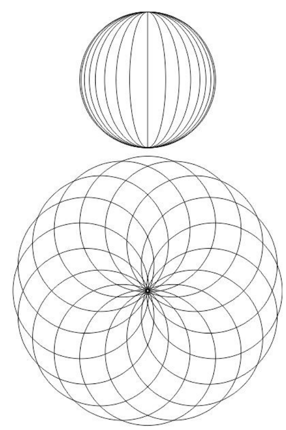


Figure 5

Symmetrical Node Puzzles for Linear A_{CL},k1

It is possible to arrange an A_{CL}k1 linear link in a symmetrical flat circle. By passing symmetrical groups of node crossings thru holes made in some bars, a puzzle like device can be made. If the holes in the bars are big enough to allow for some clearance and are spaced just right then you can rotate the bars continuously in a group about the toroid axis. When doing this rotation the circles appear to turn inside out too, but instead they just move from side to side through the torus space thereby making a nice optical illusion puzzle. The motion seems hard to imagine, especially for puzzles with more a great number of circles, n, since it looks like the circles should collide but the motion is perfectly smooth. In actuality each small segment of each circle rotates around segments of all the other circles.

Of course the 6 circle structure shown, at the left in Figure 5 (called 'Hexaka') cannot be perfectly flat when in the flat position, since the circles have thickness and pass over each other at the nodes. This works for linear links but probably would not work for a Rogue link, especially as n increases and the Rogues become more unmovable. When the bars turn 90 degrees then the figure emulates a spherical shape with open areas at the poles. The idea can be extended. Imagine that a large number of very thin $A_{CL}k1$ circles are placed closely in a flat circular array as seen at lower right in Figure 4. If some method of holding the nodes symmetrically existed this circle shape could be folded into a spherical shape as seen at top right in Figure 4. Continuing the folding would return it to a flat circle shape. The flat circle shape has a single node in the center where a concentration of all the circles pass over each other while the spherical shape has two opposed polar nodes where this happens.

Non Linear $A_{CL}k1$ or Rogue Links

A non linear or Rogue $A_{CL}k1$ cannot be manipulated so that the centers of its circles all lie on a straight line. Figure 6, presented both as a model and as a plane projection, consists of 7 circles and is an $A_{CL}k1$ Rogue link. This appears to be the smallest possible Rogue link. A rogue link also satisfies the definition of a prime where no two links can be made close. This rogue link is designated 7_{rp}. It always has the centers of its circles in a non linear arrangement.

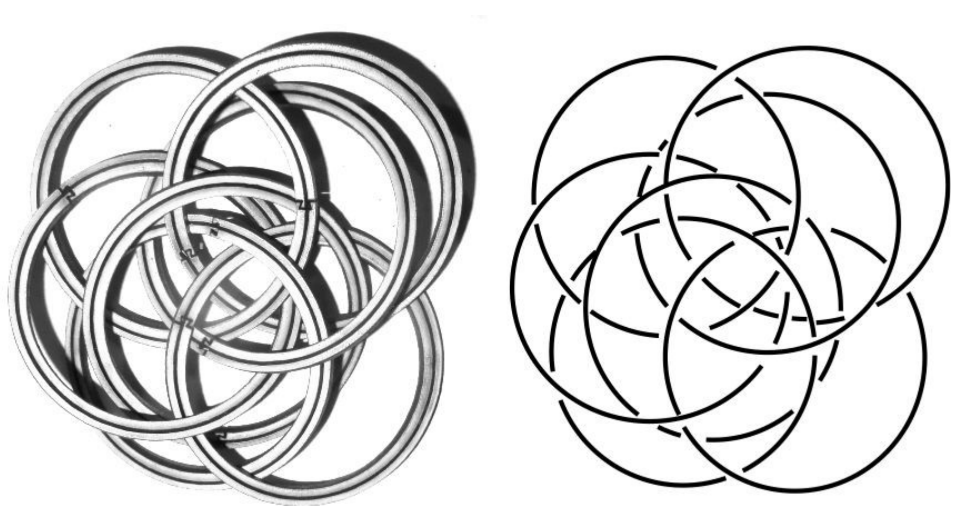


Figure 6

Here is the reasoning for why Rogue links must exist. From the above discussion about linear linkages we know that any linear prime can have up to 2n^2 unique permutation arrangements. Since n integers can be arranged left to right in n! ways this means that n!-(2n^2) of the possible attempted linear arrangements for a linear prime will not allow the centers of their circles to lie in an exact line and are candidates for making an n+1 rogue link, with this number increasing exponentially as n increases since n!-(2n^2) increases much faster than 2n^2. You can create a Rogue prime by putting a linear prime into one of these non linear attempted 'CC' arrangements. Find a position where there is no dangling naked links, then add a new link by linking it through all the circles very randomly. If this new linkage can be manipulated to form a linear prime you can try linking another new link in a different 'CC' way. Repeat this process until you get a rogue prime. The circles should be thin enough to be sure that their d/D thickness ratio does not prevent manipulation into a linear array. Once you have created a rogue prime this way way you can always make a larger one in a similar manner This method was attempted with a 5 prime but always resulted in a 6 prime not a Rogue prime. Then when it was attempted with a 6 prime a Rogue prime, 7_{rp}, was the result as shown in Figure 5. At this writing a simple construction code for Rogue primes has been developed but not considered good enough to present here.

An open question is to prove existence or nonexistence of, or to find an example of the smallest number of circles required to make a lock link. None of the circles of a lock link can be folded or laid over. A true lock link should allow d/D to become smaller and smaller and stay locked. A procedure for trying to make a lock link is to continure adding circles to a rogue link to make a more and more non linear rogue link such that the centers of the circles require 3 dimensions and cannot be laid flat.

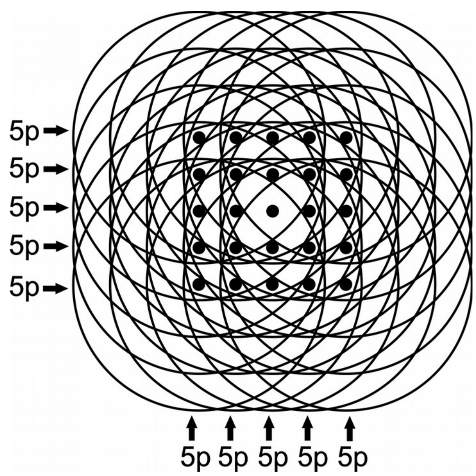


Figure 7

Circle link architecture

Figure 6 shows a possible way to produce a link that can have its circle centers lie in a plane but not in a line. It proposes to link 5 primes in a 2 dimensional grid pattern. This might result in a rogue prime. It shows another way of thinking about $A_{CL}k1$ set architectures. What kind of permutations, and twist, fold, or other properties the resulting linkage would have is unknown.

Self organizing of close links

Close links always gather together in their own groups. As the circles are made thicker this effect becomes more pronounced. The reason has to do with geometry. The close circles in a group occupy the least volume when they are close to each other. They tend to exclude links that twist the opposite way. The links in a close link all twist the same way causing like twisted circles to gather in groups and is a natural geometrical property of self organization in $A_{CL}k1$ architectures.

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